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On maximal chains in the non-crossing partition lattice

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ABSTRACT

A weak order on the set of maximal chains of the non-crossing partition lattice is introduced and studied. A 0-Hecke algebra action is used to compute the radius of the graph on these chains in which two chains are adjacent if they differ in exactly one element.

Contents

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1. Introduction

Consider the graph $G_T(n)$ with vertex set consisting of all maximal chains in $NC(n)$, the non-crossing partition lattice of type A_{n-1}, where two chains are adjacent if they differ in exactly one element. This graph may be identified with the graph of all reduced words of a given Coxeter element (long cycle) in the symmetric group S_n, where the alphabet consists of all reflections (transpositions) and two words are adjacent if they agree in all but two adjacent letters, which are (s, t) in one word and either (t^s, s) or (t, s^t) in the other; here $g^h := h^{-1}gh$. This graph is known to be connected – see, e.g., [4, Prop. 1.6.1]; we are interested in calculating its radius. This is motivated by [3] and [20], where the analogous question for simple reflections was studied, and by [15] which evaluated the radius of a related graph on labeled trees. Recall that a classical result of Hurwitz [16] implies (see [8,23]) that the number of maximal chains in the non-crossing partition lattice of type A_{n-1} is equal to the number of labeled trees on n vertices.

Our approach is to consider a 0-Hecke algebra action on the set of maximal chains. This allows us to define a well-behaved natural weak order on this set. Each maximal interval is isomorphic to the weak order on the symmetric group. The number of maximal elements, refined by a generalized inversion number, is the Carlitz–Riordan q-Catalan number. The resulting undirected Hasse diagram spans the graph $G_T(n)$ of maximal chains, implying an evaluation of the radius and an approximation of the diameter up to a factor of 3/2.

2. Basic concepts

The non-crossing partition lattice $NC(n)$ was introduced by Kreweras [17]; for a detailed discussion and generalization see [1]. It may be defined as follows.

Let T be the set of all reflections (transpositions) in the symmetric group S_n, and let $\ell_T(\cdot)$ be the corresponding length function: $\ell_T(\pi)$ is the minimal number of factors in an expression of π as a product of reflections. Let c be a Coxeter element in this group (i.e., a cycle of length n); for concreteness, take $c = (1, 2, \ldots, n)$. Then $NC(n)$ is the set

$$\left\{\pi \in S_n \colon \ \ell_T(\pi) + \ell_T\left(\pi^{-1}c\right) = \ell_T(c)\right\}$$

ordered by

$$\pi \leqslant \sigma \quad \iff \quad \ell_T(\pi) + \ell_T\left(\pi^{-1}\sigma\right) = \ell_T(\sigma).$$

Definition 2.1. Let F_n be the set of all maximal chains in the non-crossing partition lattice $NC(n)$.