J. Combin. Theory Ser. A 125 (2014) 194–213

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

# On transversals of quasialgebraic families of sets ☆

G.R. Chelnokov, V.L. Dol'nikov

*Yaroslavl State University, Sovetskaya Str. 14, Yaroslavl, 150000, Russia*

ARTICLE INFO

*Article history:*
Received 19 April 2013
Available online 28 March 2014

*Keywords:*
Helly–Gallai number
Piercing number
Motzkin's Theorem
Hadwiger–Debrunner property

ABSTRACT

The main results of this paper are generalizations of some classical theorems about transversals for families of finite sets to some cases of families of infinite sets.



## 1. Statements of the problems and the results

In this paper we consider Helly–Gallai numbers for families of sets that are similar to families of sets which are solutions for finite systems of equations.

**Definition 1.1.** A set $X$, $|X| \leqslant t$, is called a *$t$-transversal* of a family of sets $F$ if $A \cap X \neq \emptyset$ for every $A \in F$. By $\tau(F)$ denote the least positive integer $t$ such that there exists a $t$-transversal of the family $F$. This number $\tau(F)$ is called the transversal number (or piercing number) of $P$, see [11,9].

☆ The research of the authors is supported by the Russian Government project 11.G34.31.0053 and by Russian Basic Research Foundation of Fundamental Researches grant No. 13-01-00563.

*E-mail addresses:* grishabenruven@yandex.ru (G.R. Chelnokov), dolnikov@dlc.edu.yar.ru (V.L. Dol'nikov).

http://dx.doi.org/10.1016/j.jcta.2014.03.002
0097-3165/

**Definition 1.2.** The Helly–Gallai number $\mathrm{HG}(t,F)$ of a family of sets $F$ is called a minimal number $k$ such that every subfamily $G \subseteq F$ has a $t$-transversal as soon as every subfamily $H \subseteq G$ of size $|H| \leqslant k$ has a $t$-transversal, see [11,3,9].

For every family $F$, the existence of a 1-transversal is equivalent to the condition that the intersection of all sets of $F$ is nonempty. Therefore a number $\mathrm{HG}(1,F)$ is also called a Helly number $H(F)$ for a family $F$.

**Remark 1.3.** If $F$ is a family of intervals on the line, then $\mathrm{HG}(t,F) = t+1$. If $F$ is a family of convex compact sets in $\mathbb{R}^d$, $d \geqslant 2$ and $t \geqslant 2$, then $\mathrm{HG}(1,F) = d+1$ and numbers $\mathrm{HG}(t,F)$ for $t \geqslant 2$ don't exist.

The Helly numbers for a family of algebraic varieties were found by T.S. Motzkin, see [13].

**Definition 1.4.** Let $A_m^d$ be a family of sets of common zeroes in $\mathbb{R}^d$ for a finite collection of polynomials of $d$ variables and degree at most $m$.

**Theorem 1.5** *(Motzkin's Theorem). (See [13].) We have*

$$H\left(A_m^d\right) = \binom{m+d}{d}.$$

The Helly–Gallai numbers for the family of algebraic varieties $A_n^d$ were determined by M. Deza and P. Frankl [4], and V. Dol'nikov [5]. They are given by the formula:

$$\mathrm{HG}\left(t, A_m^d\right) = \binom{\binom{m+d}{d} + t - 1}{t}.$$

In the papers [5,7] the Helly–Gallai numbers for families of sets of more general kind were considered. More precisely, families were the zero sets of linear finite-dimensional subspaces of functions from a ground set $V$ to a field $\mathbb{F}$.

In particular, the Helly–Gallai numbers

$$\mathrm{HG}(t, S_{d-1}) = \binom{d+t+1}{t}$$

for families of spheres $S_{d-1}$ in $\mathbb{R}^d$ were found. Independently Helly numbers $H(S_{d-1})$ were found by H. Maehara [12].

Now we give some bounds for the Helly–Gallai numbers of quasialgebraic families of sets.

**Definition 1.6.** Let $F$ be a family of sets. Denote inductively $F^0 = F$ and

$$F^{k+1} = \left\{B\colon\ B = A_i \cap A_j,\ \text{where } A_i, A_j \in F^k \text{ and } A_i \neq A_j\right\}.$$