Journal of Combinatorial Theory, Series A 125 (2014) 273–305

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Proof of Blum's conjecture on hexagonal dungeons

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ARTICLE INFO

Article history:
Received 12 July 2013
Available online 9 April 2014

Keywords:
Tilings
Perfect matchings
Dual graph
Graphical condensation
Aztec dungeons
Hexagonal dungeons

ABSTRACT

Matt Blum conjectured that the number of tilings of the hexagonal dungeon of sides a, $2a$, b, a, $2a$, b (where $b \geqslant 2a$) is $13^{2a^2}14^{\lfloor \frac{a^2}{2} \rfloor}$ (Propp, 1999 [4]). In this paper we present a proof for this conjecture using Kuo's Graphical Condensation Theorem (Kuo, 2004 [2]).

1. Introduction

In 1999 Propp published an article [4] tracking the progress on a list of 20 open problems in the field of exact enumeration of perfect matchings, which he presented in a lecture in 1996, as part of the special program on algebraic combinatorics organized at MSRI during the academic year 1996–1997. The article also presented a list of 12 new open problems.

These 32 problems can be grouped into three broad categories: conjectures stating an explicit formula for the number of perfect matchings of the specific family of graphs they pertain to, problems for which the number of perfect matchings does not seem to be given by a simple formula, but presents some patterns that are required to be proved,

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[1] M.C. was supported in part by NSF grant DMS-1101670.

http://dx.doi.org/10.1016/j.jcta.2014.03.008

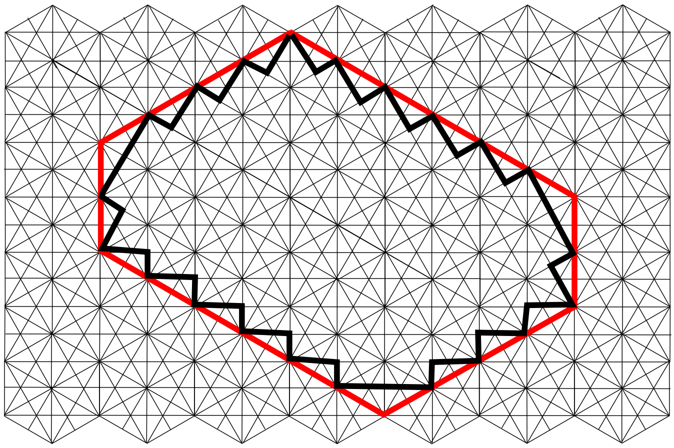

Fig. 1.1. The hexagonal dungeon of sides 2, 4, 6, 2, 4, 6 (in cyclic order, starting from the western side).

and problems concerned with various aspects of the Kasteleyn matrices of the involved graphs, and not directly with their number of perfect matchings.

In some sense, the most compelling ones to prove are the ones in the first category. The only one from the list of 12 new problems which falls into this category is Matt Blum's conjecture on the number of tilings of the so-called hexagonal dungeon regions.[2] Proving this conjecture, still open fourteen years after its publication, is the main result of the current paper.

Consider the lattice obtained from the triangular lattice by drawing in all the altitudes in all the unit triangles (i.e. the plane lattice corresponding to the affine Coxeter group G_2). On this lattice, consider a hexagonal contour of the type illustrated in Fig. 1.1. If the side-lengths of the hexagonal contour, in units equal to the side-length of the unit triangles, are a, $2a$, b, a, $2a$, b (in clockwise order, starting from the western edge), then the lattice region determined by the indicated jagged contour is called the *hexagonal dungeon of sides* a, $2a$, b, a, $2a$, b, and is denoted by $HD_{a,2a,b}$ (see Fig. 1.1 for an example). This region was introduced by Matt Blum, who discovered a striking pattern in the number of its tilings,[3] which led him to the following conjecture.

Conjecture 1 *(Matt Blum). (See Problem 25 in [4].) Assume that* a *and* b *are two positive integers so that* $b \geqslant 2a$. *Then the number of tilings of the hexagonal dungeon* $HD_{a,2a,b}$ *is* $13^{2a^2}14^{\lfloor \frac{a^2}{2} \rfloor}$.

The main result of the current paper is a proof of this conjecture. Our proof is based on Kuo's powerful graphical condensation method [2]. In order for graphical condensation to

[2] We note that there is one problem among the original 20 (Problem 16 in [4]) which stands out in a similar manner. This was solved and generalized recently by one of the authors of the current paper (T.L.); see [3].

[3] A tile is a union of two fundamental regions sharing an edge, and a tiling of a lattice region R is a covering of R by such tiles, with no gaps or overlaps (see Fig. 1.2 for an example).