Journal of Combinatorial Theory, Series A 120 (2013) 1604–1631

Contents lists available at SciVerse ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Polynomiality, wall crossings and tropical geometry of rational double Hurwitz cycles

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ARTICLE INFO

Article history:
Received 5 October 2012
Available online 13 June 2013

Keywords:
Tropical geometry
Hurwitz theory
Moduli spaces
Intersection theory

ABSTRACT

We study rational double Hurwitz cycles, i.e. loci of marked rational stable curves admitting a map to the projective line with assigned ramification profiles over two fixed branch points. Generalizing the phenomenon observed for double Hurwitz numbers, such cycles are piecewise polynomial in the entries of the special ramification; the chambers of polynomiality and wall crossings have an explicit and "modular" description. A main goal of this paper is to simultaneously carry out this investigation for the corresponding objects in tropical geometry, underlining a precise combinatorial duality between classical and tropical Hurwitz theory.

Contents

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0097-3165/$ – see front matter
http://dx.doi.org/10.1016/j.jcta.2013.05.010

1. Introduction

Hurwitz theory is the study of maps of algebraic curves, viewed as ramified covers of orientable surfaces. At the intersection of geometry, representation theory and combinatorics, it is an area that naturally lends itself to making bridges and connections. In this paper we study the combinatorial structure of certain Hurwitz spaces via a parallel investigation in tropical geometry.

1.1. Summary

The study of the connections between classical and tropical Hurwitz theory was initiated in [3] and [4]; the tropical point of view provided a combinatorial interpretation of double Hurwitz numbers which was very well tuned to describing the polynomial aspects and the wall crossing phenomena occurring in the theory. We continue this exploration by studying rational double Hurwitz loci inside spaces of (relative/tropical) stable maps, generically parameterizing covers of $\mathbb{P}^1$ with two prescribed ramification profiles and a part of the branch divisor fixed, and their pushforwards to the moduli space of curves, which we call double Hurwitz cycles. Besides the genus, the discrete invariants we fix are the total length n of the special ramification profile, and the dimension of the loci we want to study. Then we study families of Hurwitz loci parameterized by integral points in an $(n-1)$-dimensional vector space.

On the classical side, we realize Hurwitz loci as the pullback via a natural branch morphism of appropriate strata in spaces of pointed chains of projective lines (Losev–Manin spaces, Section 2.1). This gives a boundary expression for Hurwitz cycles where the coefficients are piecewise polynomials in the entries of the special ramification data (Theorem 3.3).

On the tropical side, we realize the Hurwitz loci as tropical Gromov–Witten cycles. Our tropical Hurwitz cycles are balanced polyhedral complexes; their topology is constant in the chambers of polynomiality of (classical) Hurwitz cycles, whereas their geometry (affine integral structure, weights and coordinates of vertices) varies in a polynomial way in terms of the special ramification profiles.

Naturally, we also study the connection between classical and tropical Hurwitz cycles (Section 5) and observe a natural combinatorial duality between tropical and classical strata. To be more precise, the stratification on the tropical side is the polyhedral structure inherited from the moduli space of tropical curves. The stratification on the classical side is the usual stratification in boundary classes. For Hurwitz cycles of dimension d, k-dimensional classical strata correspond to $(d-k)$-dimensional tropical strata, and the combinatorial type of the tropical stratum can be encoded in terms of the dual graph of the classical stratum.

We conclude the paper by studying how Hurwitz cycles vary across the walls of the chambers of polynomiality, both on the classical and on the tropical side. In a similar fashion to Hurwitz numbers, the wall crossing formulae have an inductive form: the cycles in the formula can be obtained as pushforwards via appropriate gluing morphism of pairs of boundary strata coming from Hurwitz cycles where the profile data is split according to the equation of the wall, and the dimensions are split in all possible ways adding up to the correct one. Even though the final form of the tropical and classical wall crossing formulae is essentially the same, there are some subtleties involved in even making sense of what a tropical wall crossing formula may be: that's why we treat the two cases separately. The classical wall crossing formula is Theorem 6.1, the (cleanest form of the) tropical one Corollary 6.5.

To make our exposition easier to follow, throughout the paper we use the 1-dimensional case ((tropical) Hurwitz curves) as a running example.