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Lattice-point generating functions for free sums of convex sets

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ABSTRACT

Let $\mathcal{J}$ and $\mathcal{K}$ be convex sets in $\mathbb{R}^n$ whose affine spans intersect at a single rational point in $\mathcal{J} \cap \mathcal{K}$, and let $\mathcal{J} \oplus \mathcal{K} = \operatorname{conv}(\mathcal{J} \cup \mathcal{K})$. We give formulas for the generating function

$$\sigma_{\operatorname{cone}(\mathcal{J}\oplus\mathcal{K})}(z_1,\ldots,z_n,z_{n+1}) = \sum_{(m_1,\ldots,m_n)\in t(\mathcal{J}\oplus\mathcal{K})\cap\mathbb{Z}^n} z_1^{m_1}\cdots z_n^{m_n} z_{n+1}^t$$

of lattice points in all integer dilates of $\mathcal{J} \oplus \mathcal{K}$ in terms of $\sigma_{\operatorname{cone}\mathcal{J}}$ and $\sigma_{\operatorname{cone}\mathcal{K}}$, under various conditions on $\mathcal{J}$ and $\mathcal{K}$. This work is motivated by (and recovers) a product formula of B. Braun for the Ehrhart series of $\mathcal{P} \oplus \mathcal{Q}$ in the case where $\mathcal{P}$ and $\mathcal{Q}$ are lattice polytopes containing the origin, one of which is reflexive. In particular, we find necessary and sufficient conditions for Braun's formula and its multivariate analogue.

1. Introduction

Given arbitrary convex subsets $\mathcal{J}, \mathcal{K} \subseteq \mathbb{R}^n$, we denote the convex hull of their union by $\mathcal{J} \oplus \mathcal{K} := \operatorname{conv}(\mathcal{J} \cup \mathcal{K})$. We call $\mathcal{J} \oplus \mathcal{K}$ a *free sum of* $\mathcal{J}$ *and* $\mathcal{K}$ when $\mathcal{J}$ and $\mathcal{K}$ each contain the origin and their respective linear spans are orthogonal coordinate subspaces (*i.e.*, subspaces spanned by subsets of the

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standard basis vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$).[1] More generally, we will write "$\mathcal{J} \oplus \mathcal{K}$ is a free sum" when $\mathcal{J} \oplus \mathcal{K}$ is a free sum of $\mathcal{J}$ and $\mathcal{K}$ up to the action of $\mathsf{SL}_n(\mathbb{Z})$ on $\mathbb{R}^n$. A familiar example is the octahedron $\operatorname{conv}\{\pm\mathbf{e}_1, \pm\mathbf{e}_2, \pm\mathbf{e}_3\}$ in $\mathbb{R}^3$, which is the free sum of the "diamond" $\operatorname{conv}\{\pm\mathbf{e}_1, \pm\mathbf{e}_2\}$ and the line segment $\operatorname{conv}\{\pm\mathbf{e}_3\}$. Free sums arise naturally in toric geometry because the free-sum operation is dual to the Cartesian product operation under polar duality: $(\mathcal{P} \times \mathcal{Q})^\vee = \mathcal{P}^\vee \oplus \mathcal{Q}^\vee$. For example, the free-sum decomposition above of the octahedron corresponds to the decomposition of the toric variety $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ as the product of $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^1$.

Our goal is to understand the integer lattice points in a free sum and its integer dilates in terms of the corresponding data for its summands. Of particular interest is the case of a free sum $\mathcal{P} \oplus \mathcal{Q}$ in which $\mathcal{P}$ and $\mathcal{Q}$ are rational polytopes. A *rational* (respectively, *lattice*) *polytope* in $\mathbb{R}^n$ is a polytope all of whose vertices are in $\mathbb{Q}^n$ (respectively, the *integer lattice* $\mathbb{Z}^n$). Given a rational polytope $\mathcal{P} \subseteq \mathbb{R}^n$, its *Ehrhart series*

$$\operatorname{Ehr}_{\mathcal{P}}(t) := 1 + \sum_{k \in \mathbb{Z}_{\geqslant 1}} \left|k\mathcal{P} \cap \mathbb{Z}^n\right| t^k$$

is the generating function of the *Ehrhart quasi-polynomial* of $\mathcal{P}$, which counts the integer lattice points in $k\mathcal{P}$ as a function of an integer dilation parameter k. Let $\operatorname{den}\mathcal{P}$ denote the *denominator* of $\mathcal{P}$, the smallest positive integer such that the corresponding dilate of $\mathcal{P}$ is a lattice polytope. A famous theorem of Ehrhart [8] says that

$$\operatorname{Ehr}_{\mathcal{P}}(t) = \frac{\delta_{\mathcal{P}}(t)}{(1 - t^{\operatorname{den}\mathcal{P}})^{\dim\mathcal{P}+1}}$$

for some polynomial $\delta_{\mathcal{P}}$, the *δ-polynomial* of $\mathcal{P}$. (Common alternative names for the δ-polynomial include *h^*-polynomial* and *Ehrhart h-vector*.) See, *e.g.*, [3,11,16] for this and many more facts about Ehrhart series.

Our work is motivated by the following result of B. Braun, which expresses the δ-polynomial of $\mathcal{P} \oplus \mathcal{Q}$ in terms of the δ-polynomials of $\mathcal{P}$ and $\mathcal{Q}$ when $\mathcal{P}$ is a reflexive polytope (defined in Section 3 below).

Theorem 1.1. *(See [4].) Suppose that $\mathcal{P}, \mathcal{Q} \subseteq \mathbb{R}^n$ are lattice polytopes such that $\mathcal{P}$ is reflexive, $\mathcal{Q}$ contains the origin in its relative interior, and $\mathcal{P} \oplus \mathcal{Q}$ is a free sum. Then*

$$\delta_{\mathcal{P}\oplus\mathcal{Q}} = \delta_{\mathcal{P}}\delta_{\mathcal{Q}}. \tag{1}$$

That is, in terms of Ehrhart series,

$$\operatorname{Ehr}_{\mathcal{P}\oplus\mathcal{Q}}(t) = (1-t)\operatorname{Ehr}_{\mathcal{P}}(t)\operatorname{Ehr}_{\mathcal{Q}}(t). \tag{2}$$

Our first main result, Theorem 1.2 below, gives a multivariate generalization of Theorem 1.1 for arbitrary compact convex sets. Our second main result, Theorem 1.3 below, characterizes the free sums of rational polytopes that satisfy our multivariate generalization of Eq. (2). A characterization of the free sums satisfying Eq. (2) itself is a consequence. Before stating our results, we first need to define some notation.

The Ehrhart series is a specialization of a multivariate Laurent series defined as follows. Let $\alpha : \mathbb{R}^n \to \mathbb{R}^{n+1}$ be the affine embedding $(a_1, \dots, a_n) \mapsto (a_1, \dots, a_n, 1)$. Given a convex set $\mathcal{K} \subseteq \mathbb{R}^n$, let $\operatorname{cone}\mathcal{K} \subseteq \mathbb{R}^{n+1}$ be the set of all nonnegative scalar multiples of elements of $\alpha(\mathcal{K})$. Equivalently, $\operatorname{cone}\mathcal{K}$ is the intersection of all linear cones containing $\alpha(\mathcal{K})$. Write $S_{\mathbb{Z}}$ for the set of integer lattice

[1] The free sum is sometimes called the *direct sum*. Diverse conditions on the summands appear in the literature. Some authors require that the origin [2], or at least a unique point of intersection [13,15], be in the interior of each summand. Others require no intersection, insisting only that the linear spans of the summands be orthogonal coordinate subspaces [4,10]. We require each summand to contain the origin, but we allow the origin to be on the boundary.