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Transversals to the convex hulls of all k-sets of discrete subsets of $\mathbb{R}^n$

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ABSTRACT

Let $k, d, \lambda \geqslant 1$ be integers with $d \geqslant \lambda$. What is the maximum positive integer n such that every set of n points in $\mathbb{R}^d$ has the property that the convex hulls of all k-sets have a transversal $(d-\lambda)$-plane? What is the minimum positive integer n such that every set of n points in general position in $\mathbb{R}^d$ has the property that the convex hulls of all k-sets do not have a transversal $(d-\lambda)$-plane? In this paper, we investigate these two questions. We define a special *Kneser hypergraph* and, by using some topological results and the well-known λ-Helly property, we relate our second question to the chromatic number of such hypergraphs. Moreover, we establish a connection (when $\lambda = 1$) with Kneser's conjecture, first proved by Lovász. Finally, we prove a discrete flat center theorem.

1. Introduction

Let A be a set of eight points in general position in $\mathbb{R}^3$. We claim that there is no transversal line to the convex hulls of all the 4-sets of A. Otherwise, if we let L be such a transversal line and $x_0 \in A$ a point not lying on L, then the plane H through x_0 and L would contain at most three points of A and so there would be at least five points of A not in H. Therefore by the pigeon-hole principle, three of these points would lie on the same side of H. Consequently the line L would not intersect the convex hull of these three points and x_0.

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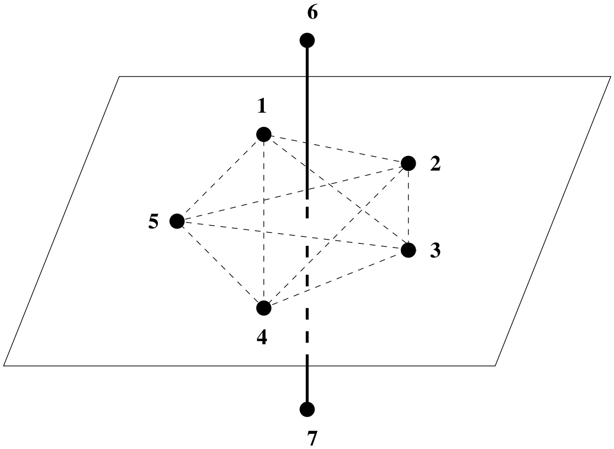

Fig. 1. $\overrightarrow{67}$ is a transversal line of all tetrahedrons.

On the other hand, if A is a set of six points in $\mathbb{R}^3$, then there is always a transversal line to the convex hulls of the 4-sets of A. For this, if $x_0 \in A$, then every 4-set either contains x_0 or is contained in $A - x_0$. Moreover, the family of 4-sets of $A - x_0$ satisfies the 3-Helly property (recall that a family F of convex sets in $\mathbb{R}^d$ has the λ-*Helly property* if every subfamily F' of F with size $\lambda + 1$ is intersecting) and consequently there is a point y_0 in the intersection of the convex hulls of these 4-sets. Therefore the line through x_0 and y_0 is a transversal line to the convex hulls of all the 4-sets of A.

With seven points in $\mathbb{R}^3$ we may have both options. The suspension of a suitable pentagon with two extra points (one above and one below the pentagon) has a transversal line to the convex hulls of the 4-sets, see Fig. 1.

The construction of a set of seven points in general position without a transversal line to the convex hulls of the 4-sets is more difficult. Such construction will be discussed at the end of the paper (see Appendix A).

We define the following two functions: let $k, d, \lambda \geqslant 1$ be integers with $d \geqslant \lambda$.

$m(k,d,\lambda) \stackrel{\text{def}}{=}$ the maximum positive integer n such that every set of n points (not necessarily in general position) in $\mathbb{R}^d$ has the property that the convex hulls of all k-sets have a transversal $(d-\lambda)$-plane,

and

$M(k,d,\lambda) \stackrel{\text{def}}{=}$ the minimum positive integer n such that for every set of n points in general position in $\mathbb{R}^d$ the convex hulls of the k-sets do not have a transversal $(d-\lambda)$-plane.

The purpose of this paper is to study the above functions. It is clear that $m(k,d,\lambda) < M(k,d,\lambda)$, and from the above we have $m(4,3,2) = 6$ and $M(4,3,2) = 8$. In the next section, we prove the following.

Theorem 1. *Let $k, d, \lambda \geqslant 1$ be integers and $d \geqslant \lambda$. Then*

$$M(k,d,\lambda) = \begin{cases} d + 2(k-\lambda) + 1 & \text{if } k \geqslant \lambda, \\ k + (d-\lambda) + 1 & \text{if } k \leqslant \lambda. \end{cases}$$

After discussing some topological results in Section 3 and following the spirit of Dol'nikov in [4] and [5], we will introduce a special *Kneser hypergraph* and establish a close connection between its