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Note

On linear forbidden submatrices

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ABSTRACT

In this paper we study the extremal problem of finding how many 1 entries an n by n 0–1 matrix can have if it does not contain certain forbidden patterns as submatrices. We call the number of 1 entries of a 0–1 matrix its weight. The extremal function of a pattern is the maximum weight of an n by n 0–1 matrix that does not contain this pattern as a submatrix. We call a pattern (a 0–1 matrix) linear if its extremal function is $O(n)$. Our main results are modest steps towards the elusive goal of characterizing linear patterns. We find novel ways to generate new linear patterns from known ones and use this to prove the linearity of some patterns. We also find the first minimal non-linear pattern of weight above 4. We also propose an infinite sequence of patterns that we conjecture to be minimal non-linear but have $\Omega(n \log n)$ as their extremal function. We prove a weaker statement only, namely that there are infinitely many minimal not quasi-linear patterns among the submatrices of these matrices. For the definition of these terms see below.

1. Introduction

The extremal theory of 0–1 matrices with respect to forbidden submatrices was initiated by the papers [1,3] more than 15 years ago. It has since attracted a lot of research. Applications to combinatorial geometry were present since the first papers, later in [7,10] this theory was applied to solve the noted Stanley–Wilf conjecture of enumerative combinatorics. This extremal theory of matrices can be considered as a Turán type extremal theory of bipartite graphs with a linear order on the vertices. See more on this connection in [11] and see [2] on the related notion of convex geometric graphs.

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1.1. Definitions

We start with the basic definitions. In this paper we consider 0–1 matrices. We consider 1 entries as representing “present” while 0 entries represent “missing”. In keeping with this we call replacing a 1 entry by 0 in a matrix *deleting* that entry. We say that the 0–1 matrix A *represents* the same size matrix B if $B = A$ or B is obtained from A by deleting several 1’s. We say that a 0–1 matrix A *contains* another 0–1 matrix B if a submatrix of A represents B. Notice that we do not allow the rows or columns to be permuted and therefore containment crucially depends on the order of the rows/columns. We say A *avoids* B if A does not contain B.

The *weight* of a 0–1 matrix P is the number of its 1 entries, denoted by $w(P)$. To avoid the trivial case of an all 0 matrix (contained in every matrix of appropriate size) we define a *pattern* to be a 0–1 matrix of weight at least 1. Our main interest is to find the *extremal function* $\mathrm{ex}(n, P)$ of the pattern P for specific patterns, where $\mathrm{ex}(n, P)$ is defined to be the maximal weight of an n by n 0–1 matrix avoiding P.

1.2. Linearity

We call a pattern P *linear* if $\mathrm{ex}(n, P) = O(n)$, otherwise P is *non-linear*. Characterizing linear patterns is of special interest but very little is known about them. Proving a conjecture of Füredi and Hajnal [4] Marcus and Tardos [10] show that permutation matrices are linear. By a result of Klazar and Valtr [9] on Davenport–Schinzel sequences certain *bitonic patterns* are also linear (see definition in Section 2 before Theorem 2.6). Beyond this only a few small patterns were shown to be linear and there were a few simple reduction rules in [4,12] that implied the linearity of certain patterns if suitable submatrices were linear. In Section 2 we establish two new reductions and use them to prove linearity of certain patterns.

We call a pattern P *minimal non-linear* if it is non-linear but all patterns $Q \neq P$ contained by P are linear. Clearly, a pattern is linear if and only if it avoids all minimal non-linear patterns.

The order of magnitude of all patterns of weight at most four was established in [4,12], so all linear and minimal non-linear patterns are known of weight at most four. However no minimal non-linear pattern has been known of larger weight and in fact finding such was raised in [12] as an open problem. In Section 3 we present a minimal non-linear pattern H_0 of weight 5. We establish that $\mathrm{ex}(n, H_0) = \Theta(n \log n)$. In fact, we give an infinite sequence of patterns H_i and we conjecture that each of them is minimal non-linear. We show that they are non-linear, moreover $\mathrm{ex}(n, H_i) = \Omega(n \log n)$ but we could not prove minimality or even that they contain infinitely many distinct minimal non-linear patterns. Instead we introduce *quasi-linearity*, a relaxation of linearity, see below, and prove a similar statement for that notion.

1.3. Quasi-linearity

We call a pattern *light* if it contains exactly one 1 entry in every column.

The close connection between the extremal function of light matrices and the Davenport–Schinzel theory of sequences was first noted in a special case by Füredi and Hajnal [4] and was developed later by Klazar. For us, the most important consequence of the connection is the following result of Klazar [7,8].

Theorem 1.1. *(See Klazar [7,8].) For any light* 0–1 *matrix A there exists a constant c such that*

$$\mathrm{ex}(n, A) \leqslant n \cdot 2^{(\alpha(n))^c}.$$

Here α is the extremely slowly growing but unbounded inverse of Ackermann’s function. As [8] is not easily accessible we include the simple deduction of this result from a fundamental result of [6] in Section 2.

The above result motivates that we call *quasi-linear* a function f if $f(n) \leqslant n \cdot 2^{(\alpha(n))^c}$ for some c. We call a pattern P *quasi-linear* if $\mathrm{ex}(n, P)$ is quasi-linear. With this terminology Theorem 1.1 states