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Asymptotic enumeration and limit laws for graphs of fixed genus

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ABSTRACT

It is shown that the number of labelled graphs with n vertices that can be embedded in the orientable surface $\mathbb{S}_g$ of genus g grows asymptotically like

$$c^{(g)} n^{5(g-1)/2-1} \gamma^n n!$$

where $c^{(g)} > 0$, and $\gamma \approx 27.23$ is the exponential growth rate of planar graphs. This generalizes the result for the planar case $g = 0$, obtained by Giménez and Noy.

An analogous result for non-orientable surfaces is obtained. In addition, it is proved that several parameters of interest behave asymptotically as in the planar case. It follows, in particular, that a random graph embeddable in $\mathbb{S}_g$ has a unique 2-connected component of linear size with high probability.

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1. Introduction and statement of main results

It has been shown by Giménez and Noy [21] that the number of planar graphs with n labelled vertices grows asymptotically as

$$c \cdot n^{-7/2}\gamma^n n!$$

where $c > 0$ and $\gamma \approx 27.23$ are well-defined analytic constants. Since planar graphs are precisely those that can be embedded in the sphere, it is natural to ask about the number of graphs that can be embedded in a given surface.

In what follows, graphs are simple and labelled with $V = \{1, 2, \dots, n\}$, so that isomorphic graphs are considered different unless they have exactly the same edges. Let $\mathbb{S}_g$ be the orientable surface of genus g, that is, a sphere with g handles, and let $a_n^{(g)}$ be the number of graphs with n vertices embeddable in $\mathbb{S}_g$. A first approximation to the magnitude of these numbers was given by McDiarmid [28], who showed that

$$\lim_{n\to\infty}\left(\frac{a_n^{(g)}}{n!}\right)^{1/n} = \gamma.$$

This establishes the *exponential growth* of the $a_n^{(g)}$, which is the same as for planar graphs and does not depend on the genus.

In this paper we provide a considerable refinement and obtain a sharp estimate, showing how the genus comes into play in the *subexponential growth*. In the next statements, γ is the exponential growth rate of planar graphs, and γ_μ is the exponential growth rate of planar graphs with n vertices and $\lfloor \mu n \rfloor$ edges. Both γ and the function γ_μ are determined analytically in [21].

Theorem 1.1. *For $g \geqslant 0$, the number $a_n^{(g)}$ of graphs with n vertices that can be embedded in the orientable surface $\mathbb{S}_g$ of genus g satisfies*

$$a_n^{(g)} \sim c^{(g)} n^{5(g-1)/2-1}\gamma^n n! \tag{1}$$

where $c^{(g)}$ is a positive constant and γ is as before.

For $\mu \in (1, 3)$, the number $a_{n,m}^{(g)}$ of graphs with n vertices and $m = \lfloor \mu n \rfloor$ edges that can be embedded in $\mathbb{S}_g$ satisfies

$$a_{n,m}^{(g)} \sim c_\mu^{(g)} n^{5g/2-4}(\gamma_\mu)^n n! \quad \text{when } n \to \infty, \tag{2}$$

where $c_\mu^{(g)}$ is a positive constant and γ_μ is as before.

We also prove an analogous result for non-orientable surfaces. Let $\mathbb{N}_h$ be the non-orientable surface of genus h, that is, a sphere with h crosscaps.

Theorem 1.2. *For $h \geqslant 1$, the number $b_n^{(h)}$ of graphs with n vertices that can be embedded in the non-orientable surface $\mathbb{N}_h$ of genus h satisfies*

$$b_n^{(h)} \sim \tilde{c}^{(h)} n^{5(h-2)/4-1}\gamma^n n! \tag{3}$$

where $\tilde{c}^{(h)}$ is a positive constant and γ is as before.

For $\mu \in (1, 3)$, the number $b_{n,m}^{(h)}$ of graphs with n vertices and $m = \lfloor \mu n \rfloor$ edges that can be embedded in $\mathbb{N}_h$ satisfies

$$b_{n,m}^{(h)} \sim \tilde{c}_\mu^{(h)} n^{5h/4-4}(\gamma_\mu)^n n! \quad \text{when } n \to \infty,$$

where $\tilde{c}_\mu^{(h)}$ is a positive constant and γ_μ is as before.