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The Eulerian distribution on involutions is indeed unimodal

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Abstract

Let $I_{n,k}$ (respectively $J_{n,k}$) be the number of involutions (respectively fixed-point free involutions) of $\{1,\dots,n\}$ with k descents. Motivated by Brenti's conjecture which states that the sequence $I_{n,0}, I_{n,1}, \dots, I_{n,n-1}$ is log-concave, we prove that the two sequences $I_{n,k}$ and $J_{2n,k}$ are unimodal in k, for all n. Furthermore, we conjecture that there are nonnegative integers $a_{n,k}$ such that

$$\sum_{k=0}^{n-1} I_{n,k}t^k = \sum_{k=0}^{\lfloor (n-1)/2\rfloor} a_{n,k}t^k(1+t)^{n-2k-1}.$$

This statement is stronger than the unimodality of $I_{n,k}$ but is also interesting in its own right.

Keywords: Involutions; Descent number; Unimodality; Eulerian polynomial; Zeilberger's algorithm

1. Introduction

A sequence $a_0, a_1, \dots, a_n$ of real numbers is said to be *unimodal* if for some $0 \leqslant j \leqslant n$ we have $a_0 \leqslant a_1 \leqslant \cdots \leqslant a_j \geqslant a_{j+1} \geqslant \cdots \geqslant a_n$, and is said to be *log-concave* if $a_i^2 \geqslant a_{i-1}a_{i+1}$ for all $1 \leqslant i \leqslant n-1$. Clearly a log-concave sequence of *positive* terms is unimodal. The reader is referred to Stanley's survey [10] for the surprisingly rich variety of methods to show that a sequence is log-concave or unimodal. As noticed by Brenti [2], even though log-concavity and unimodality have one-line definitions, to prove the unimodality or log-concavity of a sequence

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can sometimes be a very difficult task requiring the use of intricate combinatorial constructions or of refined mathematical tools.

Let $\mathfrak{S}_n$ be the set of all permutations of $[n] := \{1, \dots, n\}$. We say that a permutation $\pi = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$ has a *descent* at i $(1 \leqslant i \leqslant n-1)$ if $a_i > a_{i+1}$. The number of descents of π is called its descent number and is denoted by $\mathrm{d}(\pi)$. A statistic on $\mathfrak{S}_n$ is said to be *Eulerian*, if it is equidistributed with the descent number statistic. Recall that the polynomial

$$A_n(t) = \sum_{\pi \in \mathfrak{S}_n} t^{1+\mathrm{d}(\pi)} = \sum_{k=1}^{n} A(n,k) t^k$$

is called an *Eulerian polynomial*. It is well known that the *Eulerian numbers* $A(n,k)$ $(1 \leqslant k \leqslant n)$ form a unimodal sequence, of which several proofs have been published: such as the analytical one by showing that the polynomial $A_n(t)$ has only real zeros [3, p. 294], by induction based on the recurrence relation of $A(n,k)$ (see [9]), or by combinatorial techniques (see [7,11]).

Let $\mathcal{I}_n$ be the set of all involutions in $\mathfrak{S}_n$ and $\mathcal{J}_n$ the set of all fixed-point free involutions in $\mathfrak{S}_n$. Define

$$I_n(t) = \sum_{\pi \in \mathcal{I}_n} t^{\mathrm{d}(\pi)} = \sum_{k=0}^{n-1} I_{n,k} t^k,$$

$$J_n(t) = \sum_{\pi \in \mathcal{J}_n} t^{\mathrm{d}(\pi)} = \sum_{k=0}^{n-1} J_{n,k} t^k.$$

The first values of these polynomials are given in Table 1.

As one may notice from Table 1 that the coefficients of $I_n(t)$ and $J_n(t)$ are *symmetric* and *unimodal* for $1 \leqslant n \leqslant 6$. Actually, the symmetries had been conjectured by Dumont and were first proved by Strehl [12]. Recently, Brenti (see [5]) conjectured that the coefficients of the polynomial $I_n(t)$ are *log-concave* and Dukes [5] has obtained some partial results on the unimodality of the coefficients of $I_n(t)$ and $J_{2n}(t)$. Note that, in contrast to Eulerian polynomials $A_n(t)$, the polynomials $I_n(t)$ and $J_{2n}(t)$ may have *nonreal zeros*.

In this paper we will prove that for $n \geqslant 1$, the two sequences $I_{n,0}, I_{n,1}, \dots, I_{n,n-1}$ and $J_{2n,1}, J_{2n,2}, \dots, J_{2n,2n-1}$ are unimodal. Our starting point is the known generating functions of polynomials $I_n(t)$ and $J_n(t)$:

$$\sum_{n=0}^{\infty} I_n(t) \frac{u^n}{(1-t)^{n+1}} = \sum_{r=0}^{\infty} \frac{t^r}{(1-u)^{r+1}(1-u^2)^{r(r+1)/2}}, \tag{1.1}$$

$$\sum_{n=0}^{\infty} J_n(t) \frac{u^n}{(1-t)^{n+1}} = \sum_{r=0}^{\infty} \frac{t^r}{(1-u^2)^{r(r+1)/2}}, \tag{1.2}$$

Table 1
The polynomials $I_n(t)$ and $J_n(t)$ for $n \leqslant 6$

n	$I_n(t)$	$J_n(t)$
1	1	0
2	$1+t$	t
3	$1+2t+t^2$	0
4	$1+4t+4t^2+t^3$	$t+t^2+t^3$
5	$1+6t+12t^2+6t^3+t^4$	0
6	$1+9t+28t^2+28t^3+9t^4+t^5$	$t+3t^2+7t^3+3t^4+t^5$