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Embedding tetrahedra into quasirandom hypergraphs

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A B S T R A C T

We investigate extremal problems for quasirandom hypergraphs. We say that a 3-uniform hypergraph $H = (V, E)$ is $(d, \eta, \therefore)$-*quasirandom* if for any subset $X \subseteq V$ and every set of pairs $P \subseteq V \times V$ the number of pairs $(x, (y, z)) \in X \times P$ with $\{x, y, z\}$ being a hyperedge of H is in the interval $d|X||P| \pm \eta|V|^3$. We show that for any $\varepsilon > 0$ there exists $\eta > 0$ such that every sufficiently large $(1/2 + \varepsilon, \eta, \therefore)$-quasirandom hypergraph contains a tetrahedron, i.e., four vertices spanning all four hyperedges. A known random construction shows that the density 1/2 is best possible. This result is closely related to a question of Erdős, whether every weakly quasirandom 3-uniform hypergraph H with density bigger than 1/2, i.e., every large subset of vertices induces a hypergraph with density bigger than 1/2, contains a tetrahedron.

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1. Introduction

1.1. Extremal problems for graphs and hypergraphs

Given a fixed graph F a typical problem in extremal graph theory asks for the maximum number of edges that a (large) graph G on n vertices containing no copy of F can have. More formally, for a fixed graph F let the *extremal number* $\mathrm{ex}(n, F)$ be the number $|E|$ of edges of an F-free graph $G = (V, E)$ on $|V| = n$ vertices with the maximum number of edges. It is well known and not hard to observe that the sequence $\mathrm{ex}(n, F)/\binom{n}{2}$ is decreasing. Consequently, one may define the *Turán density*

$$\pi(F) = \lim_{n\to\infty} \frac{\mathrm{ex}(n, F)}{\binom{n}{2}},$$

which describes the maximum density of large F-free graphs. The systematic study of these extremal parameters was initiated by Turán [23], who determined $\mathrm{ex}(n, K_k)$ for complete graphs K_k. Thanks to his work and the results from [5] by Erdős and Stone it is known that the Turán density of any graph F with at least one edge can be explicitly computed using the formula

$$\pi(F) = \frac{\chi(F) - 2}{\chi(F) - 1}\,. \tag{1}$$

Already in his original work [23] Turán asked for hypergraph extensions of these extremal problems. We restrict ourselves to 3-*uniform hypergraphs* $H = (V, E)$, where $V = V(H)$ is a finite set of *vertices* and the set of *hyperedges* $E = E(H) \subseteq \binom{V}{3}$ is a collection of 3-element sets of vertices. We shall only consider graphs and 3-uniform hypergraphs and when we are referring simply to a hypergraph we will always mean a 3-uniform hypergraph. Moreover, for a simpler notation in the context of edges $\{i, j\}$ and hyperedges $\{i, j, k\}$ we omit the parentheses and just write ij or ijk. In particular, ijk denotes an unordered triple, while for ordered triples we stick to the standard notation (i, j, k).

Despite considerable effort no formula similar to (1) is known or conjectured to hold for general 3-uniform hypergraphs F. Determining the value of $\pi(F)$ is a well known and hard problem even for "simple" hypergraphs like the complete 3-uniform hypergraph $K_4^{(3)}$ on four vertices, which is also called the *tetrahedron*. Currently the best known bounds for its Turán density are

$$\frac{5}{9} \le \pi(K_4^{(3)}) \le 0.5616\,,$$

where the lower bounds is given by what is believed to be an optimal construction due to Turán (see, e.g., [3]). The upper bound is due to Razborov [13] and the proof is based on the *flag algebra method* introduced by Razborov [12]. For a thorough discussion of Turán type results and problems for hypergraphs we refer to the recent survey of Keevash [9].