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Coloring immersion-free graphs

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ABSTRACT

A graph H is *immersed* in a graph G if the vertices of H are mapped to (distinct) vertices of G, and the edges of H are mapped to paths joining the corresponding pairs of vertices of G, in such a way that the paths are pairwise edge-disjoint. The notion of an immersion is quite similar to the well-known notion of a minor, as structural approach inspired by the theory of graph minors has been extremely successful in immersions.

Hadwiger's conjecture on graph coloring, generalizing the Four Color Theorem, states that every loopless graph without a K_k-minor is $(k-1)$-colorable, where K_k is the complete graph on k vertices. This is a long standing open problem in graph theory, and it is even unknown whether it is possible to determine ck-colorability of K_k-minor-free graphs in polynomial time for some constant c. In this paper, we address coloring graphs without H-immersion. In contrast to coloring H-minor-free graphs, we show the following:

1. there exists a fixed-parameter algorithm to decide whether or not a given graph G without an immersion of a graph H of maximum degree d is $(d-1)$-colorable, where the size of H is a parameter. In fact, if G is $(d-1)$-colorable, the algorithm produces such a coloring, and

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2. for any positive integer k ($k \geq 6$), it is NP-complete to decide whether or not a given graph G without a K_k-immersion is $(k-3)$-colorable.

1. Introduction

1.1. Graph coloring and Hadwiger's conjecture

Graph coloring is arguably one of the most popular subjects in graph theory and combinatorial optimization. It is one of the hardest problems to approximate: The chromatic number is inapproximable in polynomial time within factor $n^{1-\varepsilon}$ for any $\varepsilon > 0$ unless coRP = NP [17,19]. See e.g., [21] for a survey of the problem.

The famous Four Color Theorem [2,3,33] says that every planar graph is 4-colorable. Hadwiger's conjecture [18] from 1943 suggests a far reaching generalization of the Four Color Theorem, and it is perhaps one of the most challenging open problems in graph theory. It states that every loopless graph without a K_k-minor is $(k-1)$-colorable, where K_k is the complete graph on k vertices. In 1937, Wagner [42] proved that when $k = 5$ the conjecture is in fact equivalent to the Four Color Theorem. In 1993, Robertson, Seymour and Thomas [38] proved that the case of $k = 6$ is also equivalent to the Four Color Theorem. When $k \geq 7$, Hadwiger's conjecture is still open. On the other hand, a graph without a K_k-minor is $O(k\sqrt{\log k})$-colorable for an arbitrary positive integer k [28,40]. This upper bound was proved 30 years ago, but no one improved the superlinear order $k\sqrt{\log k}$.

Hajós proposed the stronger conjecture that, for all $k \geq 1$, every graph without a subdivision of the complete graph on k vertices is $(k-1)$-colorable. Here, a graph G contains a *subdivision* of a graph H if G contains a subgraph which is isomorphic to a graph obtained from H by subdividing some edges. Certainly a subdivision of a graph H implies a minor of H, but the converse is not true. Hajós considered the conjecture in 1940s in connection with attacking the Four Color Conjecture (now Theorem). For $k \leq 4$, the conjecture is true, while, for $k = 5, 6$, it still remains open. However, for every $k \geq 7$, it was disproved by Catlin [8]. Erdős and Fajtlowicz [15] proved that the conjecture is false for almost all graphs (see also Bollobás and Catlin [6] and Thomassen [41]). In fact, the bound on the chromatic number of graphs without a K_k-subdivision is $\Theta(k^2)$ [7,27].

1.2. Immersion

In this paper, we shall consider a different graph class motivated by the theory of graph minors (see [37]). Let G, H be graphs. An *immersion* of H in G is a map η from vertices and edges of H such that