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Non-planar extensions of subdivisions of planar graphs

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Almost 4-*connectivity* is a weakening of 4-connectivity which allows for vertices of degree three. In this paper we prove the following theorem. Let G be an almost 4-connected triangle-free planar graph, and let H be an almost 4-connected non-planar graph such that H has a subgraph isomorphic to a subdivision of G. Then there exists a graph G' such that G' is isomorphic to a minor of H, and either

(i) $G' = G + uv$ for some vertices $u, v \in V(G)$ such that no facial cycle of G contains both u and v, or
(ii) $G' = G + u_1v_1 + u_2v_2$ for some distinct vertices $u_1, u_2, v_1, v_2 \in V(G)$ such that u_1, u_2, v_1, v_2 appear on some facial cycle of G in the order listed.

This is a lemma to be used in other papers. In fact, we prove a more general theorem, where we relax the connectivity assumptions, do not assume that G is planar, and consider subdivisions rather than minors. Instead of face boundaries we work with a collection of cycles that cover every edge twice and have pairwise connected intersection. Finally, we prove a version of this result that applies when $G\backslash X$ is planar for some

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set $X \subseteq V(G)$ of size at most k, but $H \backslash Y$ is non-planar for every set $Y \subseteq V(H)$ of size at most k.

1. Introduction

In this paper graphs are finite and simple (i.e., they have no loops or multiple edges). *Paths* and *cycles* have no "repeated" vertices or edges. A graph is a *subdivision* of another if the first can be obtained from the second by replacing each edge by a non-zero length path with the same ends, where the paths are disjoint, except possibly for shared ends. The replacement paths are called *segments*, and their ends are called *branch-vertices*. For later convenience a one-vertex component of a graph is also regarded as a segment, and its unique vertex as a branch-vertex. Let G, S, H be graphs such that S is a subgraph of H and is isomorphic to a subdivision of G. In that case we say that S is a *G-subdivision* in H. If G has no vertices of degree two (which will be the case in our applications), then the segments and branch-vertices of S are uniquely determined by S. An *S-path* is a path of length at least one with both ends in S and otherwise disjoint from S. A graph G is *almost 4-connected* if it is simple, 3-connected, has at least five vertices, and $V(G)$ cannot be partitioned into three sets A, B, C in such a way that $|C| = 3$, $|A| \geq 2$, $|B| \geq 2$, and no edge of G has one end in A and the other end in B.

Let a non-planar graph H have a subgraph S isomorphic to a subdivision of a planar graph G. For various problems in structural graph theory it is useful to know the minimal subgraphs of H that have a subgraph isomorphic to a subdivision of G and are non-planar. We show that under some mild connectivity assumptions these "minimal non-planar extensions" of G are quite nice:

(1.1) *Let G be an almost 4-connected planar graph on at least seven vertices, let H be an almost 4-connected non-planar graph, and let there exist a G-subdivision in H. Then there exists a G-subdivision S in H such that one of the following conditions holds:*

(i) *there exists an S-path in H joining two vertices of S not incident with the same face, or*
(ii) *there exist two disjoint S-paths with ends s_1, t_1 and s_2, t_2, respectively, such that the vertices s_1, s_2, t_1, t_2 belong to some face boundary of S in the order listed. Moreover, for $i = 1, 2$ the vertices s_i and t_i do not belong to the same segment of S, and if two segments of S include all of s_1, t_1, s_2, t_2, then those segments are vertex-disjoint.*

The connectivity assumptions guarantee that the face boundaries in a planar embedding of S are uniquely determined, and hence it makes sense to speak about incidence with faces. Theorem (1.1) is related to, but independent of [10]. We refer the reader to [13] for an overview of related results.