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Unique low rank completability of partially filled matrices

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ABSTRACT

We consider the problems of completing a low-rank positive semidefinite square matrix M or a low-rank rectangular matrix N from a given subset of their entries. We study the local and global uniqueness of such completions by analyzing the structure of the graphs determined by the positions of the known entries of M or N.

We show that, in the generic setting, the unique completability testing of rectangular matrices is a special case of the unique completability testing of positive semidefinite matrices. We prove that a generic partially filled semidefinite $n \times n$ matrix is globally uniquely rank d completable if any principal minor of size $n-1$ is locally uniquely rank d completable. These results are based on new geometric observations that extend similar results of the theory of rigid frameworks. We also give an example showing that global completability is not a generic property in $\mathbb{R}^2$.

We provide sufficient conditions for local and global unique completability of a partially filled matrix in terms of either

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the minimum number of known entries per row or the total number of known entries.

1. Introduction

We consider the problem of determining the uniqueness of a low-rank positive semidefinite completion of a partially filled matrix. This completion problem and its variants arise in various practical problems, such as computer vision, machine learning and control, and several completion algorithms have been developed and implemented in recent decades, see for example [24,21,18,16]. It is also related to the fundamental problem of Euclidean distance geometry and has been investigated from several different viewpoints, see for example [9,17].

Singer and Cucuringu [20] initiated an analysis of this problem using techniques from graph rigidity theory. They defined the underlying graph of a partially filled positive semidefinite matrix $M = (m_{ij})$ of size n as the graph G with vertex set $V = \{1, \dots, n\}$, in which ij is an edge if and only if the (i, j)-th entry (or (j, i)-th entry) is known. Note that G is *semisimple*, meaning that it has no parallel edges but may have loops.

Recall that a positive semidefinite matrix of size n and rank d can be written as $P^\top P$ for some $d \times n$ matrix P. Hence, finding a completion of M corresponds to finding a map $p : V \to \mathbb{R}^d$ such that

$$\langle p_i, p_j \rangle = m_{ij} \qquad \text{for all } ij \in E$$

where $p_i = p(i)$. Therefore, assuming that a completion is known in advance, the unique completability problem can be restated as follows. We are given a graph $G = (V, E)$ and a map $p : V \to \mathbb{R}^d$. We need to decide whether there exists a $q : V \to \mathbb{R}^d$ such that $\langle p_i, p_j \rangle = \langle q_i, q_j \rangle$ for all $ij \in E$ and $\langle p_k, p_l \rangle \neq \langle q_k, q_l \rangle$ for some $k, l \in V$.

We will adopt the terminology from rigidity theory and refer to a pair (G, p) as a *(d-dimensional) framework*. Two maps $p : V \to \mathbb{R}^d$ and $q : V \to \mathbb{R}^d$ are said to be *congruent* if

$$\langle p_i, p_j \rangle = \langle q_i, q_j \rangle \qquad \text{for all } i, j \in V \tag{1}$$

and we say that (G, q) is *equivalent* to (G, p) if

$$\langle p_i, p_j \rangle = \langle q_i, q_j \rangle \qquad \text{for all } ij \in E. \tag{2}$$

A d-dimensional framework (G, p) is called *globally uniquely completable* (or, simply, *globally completable*) in $\mathbb{R}^d$ if for every d-dimensional framework (G, q) which is equivalent