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Disjoint dijoins

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A "dijoin" in a digraph is a set of edges meeting every directed cut. D.R. Woodall conjectured in 1976 that if G is a digraph, and every directed cut of G has at least k edges, then there are k pairwise disjoint dijoins. This remains open, but a capacitated version is known to be false. In particular, A. Schrijver gave a digraph G and a subset S of its edge-set, such that every directed cut contains at least two edges in S, and yet there do not exist two disjoint dijoins included in S. In Schrijver's example, G is planar, and the subdigraph formed by the edges in S consists of three disjoint paths.

We conjecture that when $k = 2$, the disconnectedness of S is crucial: more precisely, that if G is a digraph, and $S \subseteq E(G)$ forms a connected subdigraph (as an undirected graph), and every directed cut of G contains at least two edges in S, then we can partition S into two dijoins.

We prove this in two special cases: when G is planar, and when the subdigraph formed by the edges in S is a subdivision of a caterpillar.

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1. Introduction

Some points of terminology, before we begin: in this paper, a *graph* G consists of a finite set $V(G)$ of *vertices*, a finite set $E(G)$ of *edges*, and an incidence relation between them; each edge is incident with one or two vertices (its *ends*). A *directing* of a graph G is a function η with domain $E(G)$, where $\eta(e)$ is an end of e for each $e \in E(G)$ (we call $\eta(e)$ the *head* of e). A *digraph* G consists of a graph (denoted by G^-) and a directing of G^-. If e is an edge of a graph with ends u, v, we sometimes refer to the "edge uv". If G is a digraph, and we refer to an edge uv, this means "the edge uv of G^-", and does *not* imply that this edge has head v. Our other definitions are standard.

Let G be a digraph. If $X \subseteq V(G)$, $D^+(X) = D^+_G(X)$ denotes the set of all edges of G with tail in X and head in $V(G) \setminus X$, and $D^-(X) = D^+(V(G) \setminus X)$. A *directed cut* of G means a set C of edges such that there exists $X \subseteq V(G)$ with $X, V(G) \setminus X \neq \emptyset$, and $D^-(X) = \emptyset$ and $D^+(X) = C$. A *dijoin* means a subset of $E(G)$ with nonempty intersection with every directed cut of G. D.R. Woodall [5] proposed the following conjecture in 1976:

1.1 Woodall's conjecture. *Let G be a digraph and $k \geq 0$ an integer such that every directed cut has at least k edges. Then there are k pairwise disjoint dijoins.*

This is easily proved for $k \leq 2$, but it is still open for $k = 3$, even for planar digraphs G. Interest in 1.1 stems from the Lucchesi–Younger theorem [2], which is in some sense dual:

1.2. *Let G be a digraph and $k \geq 0$ an integer such that every dijoin has at least k edges. Then there are k pairwise disjoint directed cuts.*

The Lucchesi–Younger theorem remains true in a capacitated version, as follows ($\mathbb{Z}^+$ denotes the set of non-negative integers):

1.3. *Let G be a digraph, and c a map from $E(G)$ to $\mathbb{Z}^+$, and $k \geq 0$ an integer such that $\sum_{e \in D} c(e) \geq k$ for every dijoin D. Then there are k directed cuts such that every edge e is in at most $c(e)$ of them.*

This is easily deduced from 1.2 by replacing every edge e with $c(e) > 0$ by a directed path of length $c(e)$, and by contracting every edge e with $c(e) = 0$.

However, the corresponding capacitated version of 1.1 is false; indeed, it is false even if c is $0, 1$-valued and $k = 2$ and G is planar, as an example due to A. Schrijver [3] shows. (Fig. 1.) In this paper we investigate further the case when c is $0, 1$-valued and $k = 2$.

If S, T are graphs or digraphs, we say they are *compatible* if they have the same vertex set and $E(S) \cap E(T) = \emptyset$. Thus if S, T are both graphs, then they are both subgraphs of a graph $S \cup T$, and if they are both digraphs then similarly $S \cup T$ is a digraph. Our problem in this paper is: let S, T be compatible digraphs, such that every directed cut of $S \cup T$ has at least two edges in S. When does it follow that S can be partitioned into