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On the minimum degree of minimal Ramsey graphs for multiple colours

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A B S T R A C T

A graph G is r-Ramsey for a graph H, denoted by $G \to (H)_r$, if every r-colouring of the edges of G contains a monochromatic copy of H. The graph G is called r-Ramsey-minimal for H if it is r-Ramsey for H but no proper subgraph of G possesses this property. Let $s_r(H)$ denote the smallest minimum degree of G over all graphs G that are r-Ramsey-minimal for H. The study of the parameter s_2 was initiated by Burr, Erdős, and Lovász in 1976 when they showed that for the clique $s_2(K_k) = (k-1)^2$. In this paper, we study the dependency of $s_r(K_k)$ on r and show that, under the condition that k is constant, $s_r(K_k) = r^2 \cdot \text{polylog}\, r$. We also give an

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upper bound on $s_r(K_k)$ which is polynomial in both r and k, and we show that $cr^2 \ln r \le s_r(K_3) \le Cr^2 \ln^2 r$ for some constants $c, C > 0$.

1. Introduction

A graph G is r-Ramsey for a graph H, denoted by $G \to (H)_r$, if every r-colouring of the edges of G contains a monochromatic copy of H. The fact that, for any number of colours r and every graph H, there exists a graph G such that $G \to (H)_r$ is a consequence of Ramsey's theorem [23]. Many interesting questions arise when we consider graphs G which are minimal with respect to $G \to (H)_r$. A graph G is *r-Ramsey-minimal for H* (or *r-minimal for H*) if $G \to (H)_r$, but $G' \nrightarrow (H)_r$ for any proper subgraph $G' \subsetneq G$. Let $\mathcal{M}_r(H)$ denote the family of all graphs G that are r-Ramsey-minimal with respect to H. Ramsey's theorem implies that $\mathcal{M}_r(H)$ is non-empty for all integers r and all finite graphs H. However, for general H, it is still widely open to classify the graphs in $\mathcal{M}_r(H)$, or even to prove that these graphs have certain properties.

Of particular interest is $H = K_k$, the complete graph on k vertices, and a fundamental problem is to estimate various parameters of graphs $G \in \mathcal{M}_r(K_k)$, that is, of r-Ramsey-minimal graphs for the clique on k vertices. The best-studied such parameter is the Ramsey number $R_r(H)$, the smallest number of vertices of any graph in $\mathcal{M}_r(H)$. Estimating $R_r(K_k)$, or even $R_2(K_k)$, is one of the main open problems in Ramsey theory. Classical results of Erdős [15] and Erdős and Szekeres [17] show that $2^{k/2} \le R_2(k) \le 2^{2k}$. While there have been several improvements on these bounds (see for example [10] and [28]), the constant factors in the above exponents remain the same. For multiple colours, the gap between the bounds is larger. Even for the triangle K_3, the best known upper bound on the r-colour Ramsey number $R_r(K_3)$ is of order $2^{O(r \ln r)}$ [30], whereas, from the other side, $R_r(K_3) \ge 2^{\Omega(r)}$ is the best known lower bound (see [32] for the best known constant).

Other properties of $\mathcal{M}_r(K_k)$ have also been studied: Rödl and Siggers showed in [24] that, for all $k \ge 3$ and $r \ge 2$, there exists a constant $c = c(r,k) > 0$ such that, for n large enough, there are at least 2^{cn^2} non-isomorphic graphs G on at most n vertices that are r-Ramsey-minimal for the clique K_k. In particular, $\mathcal{M}_r(K_k)$ is infinite. Another well-studied parameter is the size Ramsey number $\hat{R}_r(H)$ of a graph H, which is the minimum number of edges of a graph in $\mathcal{M}_r(K_k)$, see, e.g., [3,4,16,20,25].

Interestingly, some extremal parameters of graphs in $\mathcal{M}_r(K_k)$ could be determined exactly when the number of colours is two. In this paper, we consider the minimal minimum degree of r-Ramsey-minimal graphs $s_r(H)$, defined by