J. Combin. Theory Ser. B 116 (2016) 25–56

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series B

www.elsevier.com/locate/jctb

The structure of 2-separations of infinite matroids

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A R T I C L E I N F O

Article history:
Received 5 January 2012
Available online 29 August 2015

Keywords:
Infinite
Matroid
Tree-decomposition
Tutte
Cunningham
Edmonds
Seymour
2-separation
3-connected

A B S T R A C T

Generalizing a well known theorem for finite matroids, we prove that for every (infinite) connected matroid M there is a unique tree T such that the nodes of T correspond to minors of M that are either 3-connected or circuits or cocircuits, and the edges of T correspond to certain nested 2-separations of M. These decompositions are invariant under duality.

1. Introduction

A well known theorem of Cunningham and Edmonds [20], proved independently also by Seymour [24], states that for every connected finite matroid M there is a unique tree T such that the nodes of T correspond to minors of M each of which is either 3-connected, a circuit, or a cocircuit, and the edges of T correspond to certain 2-separations of M.

Cunningham and Edmonds also prove that, given such a decomposition tree for M with an assignment of minors and separations of M to its nodes and edges, the same

URL: http://www.math.uni-hamburg.de/home/diestel/ (R. Diestel).

http://dx.doi.org/10.1016/j.jctb.2015.05.011
0095-8956/

tree with the minors replaced by their duals defines a decomposition tree for the dual of M [20,22].

Richter [23] proved that infinite 2-connected graphs admit such a decomposition.

Our aim in this paper is to extend these results to infinite matroids, not necessarily finitary. This is less straightforward than the finite case, for two reasons. One is that we have to handle connectivity formally differently, without using rank. A more fundamental difference is that we cannot obtain the desired parts of our decomposition simply by decomposing the matroid recursively, since such a recursion might be transfinite and end with limits beyond our control. Instead, we shall define the parts explicitly, and will then have to show that they do indeed make up the entire matroid and fit together in the desired tree-structure. This is outlined in more detail in Section 2.

Our result has become possible only by the recent axiomatization of infinite matroids with duality [15]. This has already prompted a number of generalizations of standard finite matroid theorems to infinite matroids [1,3,4,2,13,5,10,9,6–8,11,12,14,16,17,19,18]. The result we prove here appears to be the first such generalization to all matroids, without any assumptions of finitariness, co-finitariness, or a combination of these.

1.1. Connectivity of infinite matroids

A matroid M is *connected* if every two of its elements lie in a common circuit. Higher-order connectivity for finite matroids is usually defined via the rank function, which is not possible for infinite matroids. However, there is a natural rank-free reformulation, as follows.

Consider a partition (X, Y) of the ground set of a matroid M, with the sets X and Y possibly empty. Given a basis B_X of $M|X$ and a basis B_Y of $M|Y$, the matroid M will be spanned by $B_X \cup B_Y$, so there exists a set $F \subseteq B_X \cup B_Y$ such that $(B_X \cup B_Y) \setminus F$ is a basis of M. Bruhn and Wollan [16] showed that the size $k = |F|$ of this set does not depend on the choices of B_X, B_Y and F, but only on (X, Y). If in addition $|X|, |Y| \geq k+1$, we call (X, Y) a *separation* of M, or more specifically a $(k+1)$-*separation*[1] or a *separation of order* $k+1$. The matroid M is n-*connected* if it has no ℓ-separation for any $\ell < n$. For M finite, these definitions are equivalent to the traditional ones.

1.2. Tree-decompositions

Let T be a tree. Consider a partition $R = (R_v)_{v \in T}$ of the ground set E of a matroid M into *parts* R_v, one for every node v of T. (We allow $R_v = \emptyset$.) Given an edge $e = vw$ of T, write T_v and T_w for the components of $T - e$ containing v and w, respectively, and put $S(e, v) := \bigcup_{u \in T_v} R_u$ and $S(e, w) := \bigcup_{u \in T_w} R_u$. If each of the partitions

[1] Some authors, including Oxley [22], call this an *exact* $(k+1)$-separation, and use the term '$(k+1)$-separation' for any separation of order at most $k+1$. The tradition of referring to $k+1$, rather than k, as the *order* of a separation with $|F| = k$ may be regrettable but is standard.