Journal of Combinatorial Theory, Series B 116 (2016) 57–86

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series B

www.elsevier.com/locate/jctb

A transition of limiting distributions of large matchings in random graphs

Pu Gao [a,1], Cristiane M. Sato [b,c,2]

[a] University of Toronto, Canada
[b] Universidade Federal do ABC, Brazil
[c] University of Waterloo, Canada

ARTICLE INFO

Article history:
Received 28 September 2013
Available online 31 August 2015

Keywords:
Matchings
Random graphs
Distribution

ABSTRACT

We study the asymptotic distribution of the number of matchings of size $\ell = \ell(n)$ in $\mathcal{G}(n,p)$ for a wide range of $p = p(n) \in (0,1)$ and for every $1 \leq \ell \leq \lfloor n/2 \rfloor$. We prove that this distribution changes from normal to log-normal as ℓ increases, and we determine the critical value of ℓ, as a function of n and p, at which the transition of the limiting distribution occurs.

1. Introduction

Let $\mathcal{G}(n,p)$ denote the probability space of random graphs on n vertices, where each edge is included independently with probability p. A classical result by Ruciński [18] shows that the distribution of the number of small subgraphs (meaning the number of subgraphs isomorphic to a graph with a fixed size) is asymptotically normal if its expected value goes to infinity as n goes to infinity. This is naturally expected as this random variable can be expressed as a sum of indicator random variables such that

E-mail addresses: pu.gao@utoronto.ca (P. Gao), cmsato@uwaterloo.ca (C.M. Sato).

[1] Research supported by the NSERC PDF (6790-404064-2011).
[2] Partially supported by FAPESP (Proc.2103/03447-6).

http://dx.doi.org/10.1016/j.jctb.2015.07.001

each variable is dependent only on a small proportion of the other variables. However, this intuitive explanation fails when the size of the subgraphs increases since then each indicator variable depends on more and more of the other variables. It has been shown by Janson [9] that the numbers of spanning trees, perfect matchings, and Hamilton cycles in $\mathcal{G}(n, p)$ (when p is in an appropriate range) are asymptotically log-normally distributed, which behave quite differently from variables with the normal distribution. The first author [5,6] recently proved that the numbers of d-factors (for d not growing too fast), triangle-factors and triangle-free subgraphs also follow a log-normal distribution (when p is in an appropriate range). Comparing the result by Ruciński [18] with that by Janson [9], we notice that the distribution of the number of ℓ-matchings (matchings of size ℓ) must undergo certain phases of transition, starting from normal and ending with log-normal, when ℓ increases from a constant size to $\lfloor n/2 \rfloor$. This motivates our research in this paper. We study the asymptotic distribution of the number of matchings of size ℓ in $\mathcal{G}(n, p)$, denoted by $X_{n,\ell}$, for every $1 \leq \ell \leq \lfloor n/2 \rfloor$. In particular, we prove that $X_{n,\ell}$ is asymptotically normal if $\ell = o(n\sqrt{p})$ and is asymptotically log-normal if $\ell = \Omega(n\sqrt{p})$. This holds for all p such that $1 - p = \Omega(1)$ and $n^{1/8-\epsilon}p \to \infty$, where $\epsilon > 0$ is an arbitrarily small constant. To our best knowledge, this is the first paper that studies the distribution of the number of copies of a subgraph whose order is between constant and n in $\mathcal{G}(n, p)$.

This same phenomenon of the transition of limiting distributions of a certain subgraph count as the size of the subgraph increases has been observed and studied in another well-known random graph space: the random d-regular graphs. There is a classical result by Bollobás [3] and Wormald [19] stating that the distributions of the numbers of short cycles (cycles with constant sizes) in a random d-regular graph are asymptotically Poisson, known as the Poisson paradigm [1], whereas it was observed later by Robinson and Wormald [16,17] that the number of Hamilton cycles is determined by the numbers of short cycles. Janson [10] proved that the logarithm of the number of Hamilton cycles can be expressed as the linear combination of a sequence of independent Poisson variables, based on the results in [16,17]. Garmo [8] filled the gap and determined the distribution of all long cycles, whose sizes vary from constant to n (i.e. the Hamiltonian cycles). His result also describes the critical point (of the size of the cycles), at which the distribution of the number of the cycles changes from a linear combination of independent Poisson variables to the exponential of that form, the same as what was described in [10].

Note that the proof of our main theorem is not just a generalisation of the proofs in [18,9,5,6]. In fact, we use very different approaches and new techniques. We do apply basic tools that also appear in [18,6] to show that a sequence of distributions converges to normal or log-normal. Our proof consists of three parts. In the first part, we study the subcritical case, where $\ell = o(n\sqrt{p})$. The second part deals with ℓ such that $\ell = \Omega(n\sqrt{p})$ but ℓ is not too close to $n/2$, whereas the last part focuses on the near-perfect matchings, where ℓ is very close to $n/2$ (i.e. $\ell = n/2 - O(n^\alpha)$ for some $0 < \alpha < 1$). The proof techniques and tools used in these three parts are different. In the first part, we will use the method of moments [11] to show that the distribution of $X_{n,\ell}$ is asymptotically