Journal of Combinatorial Theory, Series B 116 (2016) 191–207

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series B

www.elsevier.com/locate/jctb

Unavoidable tournaments

Asaf Shapira [a,1], Raphael Yuster [b]

[a] School of Mathematics, Tel-Aviv University, Tel-Aviv, 69978, Israel
[b] Department of Mathematics, University of Haifa, Haifa 31905, Israel

ARTICLE INFO

Article history:
Received 18 July 2014
Available online 10 August 2015

Keywords:
Tournament
Coloring

ABSTRACT

A basic result in Ramsey theory states that any tournament contains a "large" transitive subgraph. Since transitive tournaments contain only transitive subgraphs, it is natural to ask which subgraphs must appear in any large tournament that is "far" from being transitive. One result of this type was obtained by Fox and Sudakov who characterized the tournaments that appear in any tournament that is ϵ-far from being transitive. Another result of this type was obtained by Berger et al. who characterized the tournaments that appear in any tournament that cannot be partitioned into a bounded number of transitive sets.

In this paper we consider the common generalization of the above two results, namely the tournaments that must appear in any tournament that is ϵ-far from being the union of a bounded number of transitive sets. Our main result is a precise characterization of these tournaments.

1. Introduction

A *tournament* $T = (V, E)$ is a digraph such that for every two distinct vertices u, v exactly one of the ordered pairs (u, v) or (v, u) is an edge. A tournament is transitive

E-mail addresses: asafico@tau.ac.il (A. Shapira), raphy@math.haifa.ac.il (R. Yuster).

[1] Supported in part by ISF Grant 224/11 and a Marie-Curie CIG Grant 303320.

http://dx.doi.org/10.1016/j.jctb.2015.07.005
0095-8956/

if it contains no directed cycle, or equivalently, if it is possible to order its vertices so that all edges "point" from left to right. We use T_n to denote the (unique) n-vertex transitive tournament. If T is a tournament, we say that a subset of vertices $X \subseteq V(T)$ is *transitive* if the sub-tournament induced by X is transitive. One of the most basic results in graph theory (sometimes attributed to [19] and [12]) states that any tournament on 2^{k-1} vertices contains a transitive subset of size k (i.e., a copy of T_k). Since T_n contains only transitive subsets, it is clear that transitive tournaments are the only subgraphs that are guaranteed to appear in any tournament. It is thus natural to ask if there are any tournaments that are guaranteed to appear in any tournament that is "far" from being transitive?

Before describing the first result of this type let us introduce some definitions. We say that an n-vertex tournament T is ϵ-far from being transitive if one should change the direction of at least $\epsilon\binom{n}{2}$ of T's edges in order to turn[2] it into a transitive tournament. For a tournament H with $V(H) = \{v_1, \ldots, v_h\}$ and for a vector $(a_1, \ldots, a_h)$ of positive integers, the *transitive* $(a_1, \ldots, a_h)$*-blowup* of H is the tournament obtained by replacing each vertex v_i with the transitive tournament T_{a_i}, and connecting all edges between T_{a_i} and T_{a_j} in the same direction as the edge connecting v_i and v_j. We say that H' is a transitive blowup of H if there exists $(a_1, \ldots, a_h)$ such that H' is the transitive $(a_1, \ldots, a_h)$-blowup of H. In the case that $c = a_1 = \cdots = a_h$ we say that H' is a c-blowup. Notice that, trivially, every tournament is a transitive blowup of itself. The directed cycle on three vertices is denoted by C_3.

The first result addressing the above mentioned meta-problem was obtained by Fox and Sudakov [14] who characterized the tournaments that appear in every large enough tournament that is ϵ-far from being transitive. More precisely, let us say that a tournament H is 1-unavoidable[3] if for any $\epsilon > 0$ and $n \geq n_0(\epsilon)$, every n-vertex tournament that is ϵ-far from being transitive contains a copy of H. The result of Fox and Sudakov [14] states that a tournament H is 1-unavoidable if and only if H is either a transitive tournament or a transitive blowup of C_3.

To describe the second result we need some more definitions. For an integer $k \geq 1$, a *k-coloring* of a tournament is a partition of its vertices into k parts, where each part induces a transitive subset. The chromatic number $\chi(T)$ of a tournament T is the minimum k such that T admits a k-coloring. Berger et al. [4] call a graph H a *hero* if there is a constant c_H so that any tournament T satisfying $\chi(T) > c_H$ contains a copy of H. As noted in [7], the heroes are the tournaments that satisfy the extreme case of the well-known Erdős–Hajnal conjecture. The main result of [4] is a precise characterization of heroes (see Theorem 3 for the precise characterization).

Note that a transitive tournament T satisfies $\chi(T) = 1$ thus the 1-unavoidable tournaments studied by Fox and Sudakov [14] are those that must appear in any large enough

[2] Note that the number of edges whose direction needs to be changed in order to turn T into a transitive tournament is precisely the number of edges that need to be removed from T in order to make it cycle-free.

[3] The reason for using the 1 will become clear soon.