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Bipartite minors ☆

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ABSTRACT

We introduce a notion of bipartite minors and prove a bipartite analog of Wagner's theorem: a bipartite graph is planar if and only if it does not contain $K_{3,3}$ as a bipartite minor. Similarly, we provide a forbidden minor characterization for outerplanar graphs and forests. We then establish a recursive characterization of bipartite (2, 2)-Laman graphs — a certain family of graphs that contains all maximal bipartite planar graphs.

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1. Introduction

Wagner's celebrated theorem [5], [2, Theorem 4.4.6] provides a characterization of planar graphs in terms of minors: a graph G is planar if and only if it contains neither K_5 nor $K_{3,3}$ as a minor. Unfortunately, a minor of a bipartite graph is not always bipartite as contracting edges destroys 2-colorability. Here, we introduce a notion of a *bipartite minor*: an operation that applies to bipartite graphs and outputs bipartite graphs. We then prove a bipartite analog of Wagner's theorem: a bipartite graph is planar if and only if it does not contain $K_{3,3}$ as a bipartite minor. Similarly, we provide a forbidden bipartite minor characterization for bipartite outerplanar graphs and forests.

All the graphs considered in this note are simple graphs. A graph with vertex set V and edge set E is denoted by $G = (V, E)$. We denote the edge connecting vertices i and j by ij. A graph is *bipartite* if there exists a bipartition (or bicoloring in red and blue) of the vertex set V of G, $V = A \uplus B$, in such a way that no two vertices from the same part are connected by an edge. When discussing bipartite graphs, we fix such a bipartition and write $G = (A \uplus B, E)$; we refer to A and B as *parts* or *sides* of G.

As bipartite planar graphs with $n \geq 3$ vertices have at most $2n - 4$ edges, and as all their subgraphs are also bipartite and planar, and hence satisfy the same restriction on the number of edges, it is natural to consider the family of maximal bipartite graphs possessing this property. Specifically, we say that a bipartite graph $G = (A \uplus B, E)$ with $|A| \geq 2$ and $|B| \geq 2$ is $(2, 2)$-*Laman* if (i) G has exactly $2(|A| + |B|) - 4$ edges, and (ii) every subgraph H of G with at least 3 vertices has at most $2|V(H)| - 4$ edges. Note that the family of $(2, 2)$-Laman graphs is strictly larger than that of maximal bipartite planar graphs: indeed, taking $n \geq 2$ copies of $K_{3,3}$ minus an edge, and gluing all these copies together along the two vertices of the missing edge, produces a graph on $4n + 2$ vertices with $8n$ edges; this graph is $(2, 2)$-Laman, but it is not planar.

Our second main result is a recursive characterization of $(2, 2)$-Laman graphs. We remark that the name $(2, 2)$-Laman is motivated by Laman's theorem [4] from rigidity theory of graphs, and its relation to a recent theory of rigidity for bipartite graphs can be found in [3]. As such, this paper is a part of a project to understand notions of minors and graph-rigidity for bipartite graphs as well as to understand higher-dimensional generalizations.

The rest of this note is organized as follows: in Section 2 we define bipartite minors and prove the bipartite analog of Wagner's theorem and analogous theorems for bipartite outerplanar graphs and forests (deferring treatment of some of the cases to Appendix A). Then in Section 3 we discuss $(2, 2)$-Laman graphs.

2. Wagner's theorem for bipartite graphs

We start by defining a couple of basic operations on (bipartite) graphs. If $G = (V, E)$ is a graph and v is a vertex of G, then $G - v$ denotes the induced subgraph of G on the vertex set $V - \{v\}$. If $G = (V = A \uplus B, E)$ is a bipartite graph and u, v are two vertices