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Perfect matchings in 4-uniform hypergraphs

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ABSTRACT

A perfect matching in a 4-uniform hypergraph on n vertices is a subset of $\lfloor \frac{n}{4} \rfloor$ disjoint edges. We prove that if H is a sufficiently large 4-uniform hypergraph on $n = 4k$ vertices such that every vertex belongs to more than $\binom{n-1}{3} - \binom{3n/4}{3}$ edges, then H contains a perfect matching. A construction due to Hàn, Person, and Schacht shows that this result is the best possible.

1. Introduction and notation

For graphs we follow the notation in [3]. For a set T, we refer to all of its k-element subsets (k-sets for short) as $\binom{T}{k}$ and to the number of such k-sets as $\binom{|T|}{k}$. We say that $H = (V(H), E(H))$ is an r-uniform hypergraph or r-graph for short, where $V(H)$ is the set of vertices and $E \subset \binom{V(H)}{r}$ is a family of r-sets of $V(H)$. When the graph referred to is clear from the context we will use V instead of $V(H)$ and will identify H with $E(H)$. For an r-graph H and a set $D = \{v_1, \ldots, v_d\} \in \binom{V}{d}, 1 \leq d \leq r$, the degree of D in H, $deg_H(D) = deg_r(D)$ denotes the number of edges of H that contain D. For $1 \leq d \leq r$, let

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$$\delta_d = \delta_d(H) = \min\left\{ deg_r(D) \ : \ D \in \binom{V}{d} \right\}.$$

We say that $H(V_1, \ldots, V_r)$ is an r-partite r-graph if there is a partition of $V(H)$ into r sets, i.e., $V(H) = V_1 \cup \cdots \cup V_r$, and every edge of H uses exactly one vertex from each V_i. We call it a balanced r-partite r-graph if all V_i's are of the same size. Furthermore, $H(V_1, \ldots, V_r)$ is a complete r-partite r-graph if every r-tuple that uses one vertex from each V_i belongs to $E(H)$. We denote a balanced complete r-partite r-graph by $K^{(r)}(t)$, where $t = |V_i|$. For $r = 3$, we refer to the balanced 3-partite 3-graph $H(V_1, V_2, V_3)$, where $|V_i| = 4$ as a $4 \times 4 \times 4$ 3-graph.

When A and B are disjoint subsets of $V(H)$ of an r-graph H, for a vertex $v \in A$ we denote by $deg_r\left(v, \binom{B}{r-1}\right)$ the number of $(r-1)$-sets of B that make edges with v, while $e_r\left(A, \binom{B}{r-1}\right)$ is the sum of $deg_r\left(v, \binom{B}{r-1}\right)$ over all $v \in A$ and $d_r\left(A, \binom{B}{r-1}\right) = e_r\left(A, \binom{B}{r-1}\right) / |A|\binom{|B|}{r-1}$. We denote by $H\left(A, \binom{B}{r-1}\right)$ such an r-graph when all edges use one vertex from A and $r-1$ vertices from B. Similarly $H\left(A, B, \binom{C}{r-2}\right)$ is an r-graph where A, B, and C are disjoint subsets of $V(H)$ and every edge in H uses one vertex each from A and B and $r-2$ vertices from C. Degree of a vertex $v \in A$ is the number of $(r-1)$-sets of $V(H)$, containing one vertex from B and $r-2$ vertices from C that make edges with v. Degree of a vertex in B is similarly defined. $e_r\left(A, B, \binom{C}{r-2}\right)$ is the number of edges in $H\left(A, B, \binom{C}{r-2}\right)$, while its density is

$$d_r\left(A, B, \binom{C}{r-2}\right) = \frac{e_r\left(A, B, \binom{C}{r-2}\right)}{|A||B|\binom{|C|}{r-2}}.$$

When A and B are disjoint subsets of $V(H)$, for $1 < k < r-1$, $H\left(\binom{A}{k}, \binom{B}{r-k}\right)$ is an r-graph where all edges use k vertices from A and $r-k$ vertices from B. $e_r\left(\binom{A}{k}, \binom{B}{r-k}\right)$ is similarly defined as above. When $A_1, \ldots, A_r$ are disjoint subsets of $V(H)$, for a vertex $v \in A_1$ we denote by $deg_r(v, (A_2 \times \cdots \times A_r))$ the number of edges in the r-partite r-graph induced by subsets $\{v\}, A_2, \ldots, A_r$, and $e_r(A_1, (A_2 \times \cdots \times A_r))$ is the sum of $deg_r(v, (A_2 \times \cdots \times A_r))$ over all $v \in A_1$. Similarly

$$d_r(A_1, (A_2 \times \cdots \times A_r)) = \frac{e_r(A_1, (A_2 \times \cdots \times A_r))}{|A_1 \times A_2 \times \cdots \times A_r|}$$

An r-graph H on n vertices is η-*dense* if it has at least $\eta\binom{n}{r}$ edges. We use the notation $d_r(H) \geq \eta$ to refer to an η-*dense* r-graph H. A bipartite graph $G = (A, B)$ is η-dense if $d(A, B) = e(A, B)/(|A||B|) \geq \eta$. For $U \subset V$, $H|_U$ is the restriction of H to U. For simplicity we refer to $d_r(H|_U)$ as $d_r(U)$ and to $E(H|_U)$ as $E(U)$. A matching in H is a set of disjoint edges of H, and a perfect matching is a matching that contains all vertices. We will only deal with r-graphs on n vertices where $n = rk$ for some integer k; we denote