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On the chromatic number of random regular graphs [☆]

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A R T I C L E I N F O

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A B S T R A C T

Let $G(n, d)$ be the random d-regular graph on n vertices. For every integer k exceeding a certain constant k_0 we identify a number $d_{k\text{-col}}$ such that $G(n, d)$ is k-colorable w.h.p. if $d < d_{k\text{-col}}$ and non-k-colorable w.h.p. if $d > d_{k\text{-col}}$.

1. Introduction

Let $G(n, d)$ be the random d-regular graph on the vertex set $V = \{1, \ldots, n\}$. Unless specified otherwise, we let d and $k \geq 3$ be n-independent integers. In addition, we let $G_{\mathrm{ER}}(n, m)$ denote the uniformly random graph on V with precisely m edges (the "Erdős–Rényi model"). We say that a property $\mathcal{E}$ holds ***with high probability*** ('w.h.p.') if $\lim_{n\to\infty} \mathrm{P}[\mathcal{E}] = 1$.

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1.1. Results

Determining the chromatic number of random graphs is one of the longest-standing challenges in probabilistic combinatorics. For the Erdős–Rényi model, the single most intensely studied model in the random graphs literature, the question dates back to the seminal 1960 paper that started the theory of random graphs [18].[2] Apart from $G_{\mathrm{ER}}(n, m)$, the model that has received the most attention certainly is the random regular graph $G(n, d)$ [10,22]. In the present paper, we provide an almost complete solution to the chromatic number problem on $G(n, d)$, at least in the case that d remains fixed as $n \to \infty$ (which we regard as the most interesting regime).

The strongest previous result on the chromatic number of $G(n, d)$ is due to Kemkes, Pérez-Giménez and Wormald [23]. They proved that w.h.p. for $k \geq 3$

$$\chi(G(n,d)) = k \ \text{ if } \ d \in ((2k-3)\ln(k-1), (2k-2)\ln(k-1)), \ \text{and} \tag{1.1}$$

$$\chi(G(n,d)) \in \{k, k+1\} \ \text{ if } \ d \in [(2k-2)\ln(k-1), (2k-1)\ln k]. \tag{1.2}$$

These bounds imply that $G(n, d)$ is k-colorable w.h.p. if $d < (2k-2)\ln(k-1)$, while $G(n, d)$ fails to be k-colorable w.h.p. if $d > (2k-1)\ln k$. The main result of the present paper is

Theorem 1.1. *There is a sequence $(\varepsilon_k)_{k\geq 3}$ with $\lim_{k\to\infty} \varepsilon_k = 0$ such that the following is true.*

1. *If $d \leq (2k-1)\ln k - 2\ln 2 - \varepsilon_k$, then $G(n, d)$ is k-colorable w.h.p.*
2. *If $d \geq (2k-1)\ln k - 1 + \varepsilon_k$, then $G(n, d)$ fails to be k-colorable w.h.p.*

We have not attempted to explicitly extract or even optimize the error term ε_k.

Theorem 1.1 implies the following "threshold result".

Corollary 1.2. *There is a constant $k_0 > 0$ such that for any integer $k \geq k_0$ there exists a number $d_{k\text{-col}}$ with the following two properties.*

- *If $d < d_{k\text{-col}}$, then $G(n, d)$ is k-colorable w.h.p.*
- *If $d > d_{k\text{-col}}$, then $G(n, d)$ fails to be k-colorable w.h.p.*

To obtain Corollary 1.2, let ε_k as in Theorem 1.1 and consider the interval

$$I_k = ((2k-1)\ln k - 2\ln 2 - \varepsilon_k, (2k-1)\ln k - 1 + \varepsilon_k).$$

[2] The chromatic number problems on $G_{\mathrm{ER}}(n, m)$ and on the binomial random graph (where each pair of vertices is connected with probability $p = m/\binom{n}{2}$ independently) turn out to be equivalent [22, Chapter 1].