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On a symmetric representation of Hermitian matrices and its applications to graph theory

Masanori Sawa [1]

Kobe University, Japan

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A B S T R A C T

We give an inequality on the inertia of Hermitian matrices with some symmetry and discuss algebraic conditions for equality. The basic results also have various applications in the theories of graph decompositions, graph embeddings, and block designs.

1. Introduction

A classical problem in graph theory and discrete geometry is the isometric embedding of a finite connected graph G into a premetric space $\{0, 1, *\}^N$, called a (binary) squashed hypercube. In [9], Witsenhausen proved that if G is isometrically embedded in a premetric space $\{0, 1, *\}^N$, then the dimension N is bounded from below by the maximum of the number of positive and negative eigenvalues of the distance matrix of G. An *eigensharp graph*, a graph with equality in Witsenhausen bound, has been extensively studied by many researchers in graph theory, discrete geometry, and other related areas; for example, see [9,11,15,26] and the references therein. Recently, this result has been naturally generalized for q-ary squashed hypercubes in [23].

E-mail address: sawa@people.kobe-u.ac.jp.

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The isometric embedding of graphs into squashed hypercubes is closely related to a certain graph decomposition problem. A famous theorem by Graham and Pollak [10] asserts that if the complete graph K_n of order n is the union of edge-disjoint complete bipartite subgraphs that are not necessarily isomorphic, then the number of bipartite subgraphs is no less than $n - 1$. Liu and Schwenk [17] showed a q-analog of the Graham–Pollak theorem, namely, they proved that if the complete graph K_n is decomposed into edge-disjoint complete q-partite subgraphs, then the number of subgraphs is bounded from below by $(n - 1)/(q - 1)$. A full generalization of the Liu–Schwenk theorem was established by Gregory and vander Meulen [12] for general graphs.

On the other hand, as implied by van Lint and Wilson [16, p. 433], the Graham–Pollak theorem brings to mind a well-known theorem by De Bruijn and Erdös [4], which asserts, in a quite different terminology in incidence geometry, that if the complete graph K_n is the union of edge-disjoint subgraphs each of which is isomorphic to a given complete subgraph, then the number of subgraphs is greater than or equal to n. The De Bruijn–Erdös theorem can be regarded as a special case of Fisher's inequality [7] in design of experiments.

The main aim of this paper is to provide a systematic treatment of the three topics mentioned above in an algebraic manner. The paper is organized as follows. In Section 2, we give two inequalities on the inertia of a Hermitian matrix with some "symmetry"; see Proposition 2.1 and Theorem 2.1. Our bounds are a kind of spectral inequalities and provide a q-analog of Witsenhausen's inequality and Graham–Pollak inequality. Proposition 2.1, which is a generalization of work in [13], also gives algebraic conditions for equality that are used for further arguments in Section 3. We enjoy two different proofs of the former inequality, to provide a new geometric insight of Witsenhausen's inequality and Graham–Pollak inequality. We also discuss how we can get a "symmetric" representation of a Hermitian matrix; for the details, see Subsection 2.3. In Section 3, Witsenhausen's inequality, Graham–Pollak inequality and Fisher's inequality are rephrased in a matrix form, respectively. We reprove them using the basic results of Section 2, and discuss their generalizations, together with some new related results. In Section 4 brief remarks will be made, in connection with a question by Graham and Lovász [8] on distance matrices of graphs as well as a problem posed by Colbourn and Rosa [6] on a set of Steiner triple systems with a certain "hierarchy structure".

2. Basic results

The famous spectral inequality due to A. Hirsch asserts that the modulus of an eigenvalue of an $n \times n$ $\mathbb{C}$-valued matrix A is less than n multiplied by the maximum of the entries of A. There are a number of similar spectral inequalities for positive definite matrices. In this section, we give a quite different type of spectral bounds together with two different proofs, and discuss algebraic conditions for equality. An improvement of this bound is also given.