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# The automorphisms of bi-Cayley graphs ☆

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ABSTRACT

A bi-Cayley graph $\Gamma$ is a graph which admits a semiregular group $H$ of automorphisms with two orbits. In this paper, the normalizer of $H$ in the full automorphism group of $\Gamma$ is determined. Applying this, a characterization of cubic edge-transitive graphs of order a 2-power is given. As byproducts, we answer a problem proposed in Godsil (1983) [16] regarding the existence of arc-regular non-normal Cayley graphs of order a 2-power, and construct the first known family of cubic semisymmetric graphs of order a 2-power.



## 1. Introduction

Throughout this paper we denote by $\mathbb{Z}_n$ the cyclic group of order $n$. For a group $G$, the center and the derived subgroup of $G$ will be represented by $Z(G)$ and $G'$, respectively, and $\mathrm{Aut}(G)$ denotes the automorphism group of $G$. All graphs considered in this paper are finite, connected, simple and undirected. Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a graph with vertex set $V(\Gamma)$, and edge set $E(\Gamma)$. We use $\mathrm{Aut}(\Gamma)$ to denote its full automorphism group. For

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the group-theoretic and graph-theoretic terminology not defined here we refer the reader to [5,34].

Let $G$ be a permutation group on a set $\Omega$ and $\alpha \in \Omega$. Denote by $G_\alpha$ the stabilizer of $\alpha$ in $G$, that is, the subgroup of $G$ fixing the point $\alpha$. We say that $G$ is *semiregular* on $\Omega$ if $G_\alpha = 1$ for every $\alpha \in \Omega$ and *regular* if $G$ is transitive and semiregular. It is well-known that a graph $\Gamma$ is a *Cayley graph* if it has a group of automorphisms acting regularly on $V(\Gamma)$ (see [3, Lemma 16.3]). If we, instead, require the graph $\Gamma$ having a group of automorphisms acting semiregularly on $V(\Gamma)$ with two orbits, then we obtain a so-called *bi-Cayley graph*. (Some authors have used the term *semi-Cayley* [14,13,18,7]. In this paper, we follow [17] and use the term bi-Cayley.) Many properties of bi-Cayley graphs have been studied by researchers from various classical fields of graph theory such as strongly regular graphs [18,24,27,7], $n$-extendable graphs [13,23], the spectrum of graphs [14], Hamiltonian graphs [33], the connectivity of graphs [6,20]. We refer the reader to [1,2,26,25] for some more recent studies on bi-Cayley graphs.

Bi-Cayley graphs also play an important role in the study of symmetry in graphs. For example, by using bi-Cayley graphs, several infinite families of semisymmetric graphs were constructed in [8,9,22]. Bi-Cayley graphs can also be used to construct non-Cayley vertex-transitive graphs, and the typical examples are the generalized Petersen graphs which are bi-Cayley graphs over cyclic groups.

When dealing with the symmetry of graphs, the goal is to gain as much information as possible about the group of automorphisms of the graph. Thus it will be useful to investigate the automorphisms of Cayley graphs and bi-Cayley graphs. Cayley graphs are usually defined in the following way. Given a finite group $G$ and an inverse closed subset $S \subseteq G \setminus \{1\}$, the *Cayley graph* $\mathrm{Cay}(G,S)$ on $G$ with respect to $S$ is a graph with vertex set $G$ and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. For any $g \in G$, $R(g)$ is the permutation of $G$ defined by $R(g) : x \mapsto xg$ for $x \in G$. Set $R(G) := \{R(g) \mid g \in G\}$. It is well-known that $R(G)$ is a subgroup of $\mathrm{Aut}(\mathrm{Cay}(G,S))$. In 1981, Godsil [15] proved that the normalizer of $R(G)$ in $\mathrm{Aut}(\mathrm{Cay}(G,S))$ is $R(G) \rtimes \mathrm{Aut}(G,S)$, where $\mathrm{Aut}(G,S)$ is the group of automorphisms of $G$ fixing the set $S$ set-wise. This result has been successfully used in characterizing various families of Cayley graphs $\mathrm{Cay}(G,S)$ such that $R(G) = \mathrm{Aut}(\mathrm{Cay}(G,S))$ (see, for example, [15,16]). A Cayley graph $\mathrm{Cay}(G,S)$ is said to be *normal* if $R(G)$ is normal in $\mathrm{Aut}(\mathrm{Cay}(G,S))$. This concept was introduced by Xu in [35], and for more results about normal Cayley graphs, we refer the reader to [12].

In this paper, we shall be concerned with the automorphism groups of bi-Cayley graphs. We first note that every bi-Cayley graph admits the following concrete realization. Let $R$, $L$ and $S$ be subsets of a group $H$ such that $R = R^{-1}$, $L = L^{-1}$ and $R \cup L$ does not contain the identity element of $H$. Define the graph $\mathrm{BiCay}(H,R,L,S)$ to have vertex set the union of the *right part* $H_0 = \{h_0 \mid h \in H\}$ and the *left part* $H_1 = \{h_1 \mid h \in H\}$, and edge set the union of the *right edges* $\{\{h_0, g_0\} \mid gh^{-1} \in R\}$, the *left edges* $\{\{h_1, g_1\} \mid gh^{-1} \in L\}$ and the *spokes* $\{\{h_0, g_1\} \mid gh^{-1} \in S\}$. For the case when $|S| = 1$, the bi-Cayley graph $\mathrm{BiCay}(H,R,L,S)$ is also called *one-matching bi-Cayley graph* (see [17]).