Journal of Combinatorial Theory, Series B 118 (2016) 1–12

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series B

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Unavoidable induced subgraphs in large graphs with no homogeneous sets ☆

Maria Chudnovsky [a], Ringi Kim [a], Sang-il Oum [b], Paul Seymour [a]

[a] *Princeton University, Princeton, NJ 08544, USA*
[b] *Department of Mathematical Sciences, KAIST, Daejeon, 34141 South Korea*

ARTICLE INFO

Article history:
Received 21 April 2015
Available online 10 February 2016

Keywords:
Modular decomposition
Induced subgraph
Prime graph
Ramsey

ABSTRACT

A *homogeneous* set of an n-vertex graph is a set X of vertices ($2 \leq |X| \leq n-1$) such that every vertex not in X is either complete or anticomplete to X. A graph is called *prime* if it has no homogeneous set. A *chain* of length t is a sequence of $t+1$ vertices such that for every vertex in the sequence except the first one, its immediate predecessor is its unique neighbor or its unique non-neighbor among all of its predecessors. We prove that for all n, there exists N such that every prime graph with at least N vertices contains one of the following graphs or their complements as an induced subgraph: (1) the graph obtained from $K_{1,n}$ by subdividing every edge once, (2) the line graph of $K_{2,n}$, (3) the line graph of the graph in (1), (4) the half-graph of height n, (5) a prime graph induced by a chain of length n, (6) two particular graphs obtained from the half-graph of height n by making one side a clique and adding one vertex.

☆ M.C. is supported by NSF grant DMS-1265803. S.O. is supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Science, ICT and Future Planning (2011-0011653). P.S. is supported by ONR grant N00014-10-1-0680 and NSF grant DMS-1265563.

E-mail addresses: mchudnov@math.princeton.edu (M. Chudnovsky), ringikim@princeton.edu (R. Kim), sangil@kaist.edu (S. Oum), pds@math.princeton.edu (P. Seymour).

http://dx.doi.org/10.1016/j.jctb.2016.01.008
0095-8956/

1. Introduction

All graphs in this paper are simple and undirected. We wish to prove a theorem analogous to the following theorems. (For missing definitions in the following, please refer to the referenced papers.)

Ramsey's Theorem: *Every sufficiently large graph contains* K_n *or* $\overline{K_n}$ *as an induced subgraph.*

Folklore; see Diestel [1, Proposition 9.4.1]: *Every sufficiently large* ***connected*** *graph contains* K_n, $K_{1,n}$, *or a path of length* n *as an induced subgraph.*

Folklore; see Diestel [1, Proposition 9.4.2]: *Every sufficiently large* ***2-connected*** *graph contains* C_n *or* $K_{2,n}$ *as a topological minor.*

Oporowski, Oxley, and Thomas [6]: *Every sufficiently large* ***3-connected*** *graph contains the* n*-spoke wheel or* $K_{3,n}$ *as a minor.*

Oporowski, Oxley, and Thomas [6]: *Every sufficiently large* ***internally 4-connected*** *graph contains the* $2n$*-spoke double wheel, the* n*-rung circular ladder, the* n*-rung Möbius ladder, or* $K_{4,n}$ *as a minor.*

Ding, Chen [2]: *Every sufficiently large* ***connected and anticonnected*** *graph contains one of the following graphs or their complements as an induced subgraph: a path of length* n*, the graph obtained from* $K_{1,n}$ *by subdividing an edge once,* $K_{2,n}$ *minus one edge, or the graph obtained from* $K_{2,n}$ *by adding an edge between two degree-*n *vertices* x_1 *and* x_2 *and adding a pendant edge at each* x_i*.*

Kwon, Oum [3]: *Every sufficiently large graph with* ***no non-trivial split*** *contains, as a vertex-minor, a cycle of length* n *or the line graph of* $K_{2,n}$*.*

These results state that every sufficiently large graph satisfying certain connectivity requirements contains at least one of the given graphs. Furthermore, in all these theorems, the list is best possible in the sense that each given graph satisfies the required connectivity, can grow its size arbitrary, and does not contain other given graphs in the list.

In this paper, we focus on graphs with no homogeneous sets. A set X of vertices of a graph G is *homogeneous* if $2 \leq |X| < |V(G)|$ and every vertex outside of X is either adjacent to all vertices in X or adjacent to no vertex in X. In the literature, homogeneous sets are also called *non-trivial modules*, *partitive sets*, *autonomous sets*, and various other terms [5]. A graph is called *prime* if it has no homogeneous set.

Homogeneous sets are widely used as a tool to study classes of graphs with respect to the induced subgraph relation. If a graph G is obtained from some non-trivial graph by substituting a non-trivial graph for a vertex, then G has a homogeneous set and so G is not prime. (A graph is *non-trivial* if it has at least two vertices.) Thus, if a certain class $\mathcal{C}$ of graphs is closed under substitution and a graph H is not in $\mathcal{C}$ but all proper induced subgraphs of H are in $\mathcal{C}$, then H is prime. Many important graph classes are