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Induced subgraphs of graphs with large chromatic number. II. Three steps towards Gyárfás' conjectures

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Gyárfás conjectured in 1985 that for all k, ℓ, every graph with no clique of size more than k and no odd hole of length more than ℓ has chromatic number bounded by a function of k, ℓ. We prove three weaker statements:

- Every triangle-free graph with sufficiently large chromatic number has an odd hole of length different from five;
- For all ℓ, every triangle-free graph with sufficiently large chromatic number contains either a 5-hole or an odd hole of length more than ℓ;
- For all k, ℓ, every graph with no clique of size more than k and sufficiently large chromatic number contains either a 5-hole or a hole of length more than ℓ.

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1. Introduction

All graphs in this paper are finite, and without loops or parallel edges. A *hole* in a graph G is an induced subgraph which is a cycle of length at least four, and an *odd hole* means a hole of odd length. (The *length* of a path or cycle is the number of edges in it, and we sometimes call a hole of length n an *n-hole*.) In 1985, A. Gyárfás [2] made a sequence of three famous conjectures:

1.1. Conjecture. *For every integer k there exists $n(k)$ such that every graph G with no clique of cardinality more than k and no odd hole has chromatic number at most $n(k)$.*

1.2. Conjecture. *For all integers k, ℓ there exists $n(k, \ell)$ such that every graph G with no clique of cardinality more than k and no hole of length more than ℓ has chromatic number at most $n(k, \ell)$.*

1.3. Conjecture. *For all integers k, ℓ there exists $n(k, \ell)$ such that every graph G with no clique of cardinality more than k and no odd hole of length more than ℓ has chromatic number at most $n(k, \ell)$.*

In a recent paper [4], two of us proved the first conjecture. Note that the first two conjectures are special cases of the third. In the case of the third conjecture, we might as well assume that $k \geq 2$, and $\ell \geq 3$ and is odd. Thus it follows from [4] that Conjecture 1.3 holds for all pairs (k, ℓ) when $\ell = 3$. No other cases have been settled at the time of writing this paper, and the cases when $k = 2$ are presumably the simplest to attack next. Here we settle the first open case, when $k = 2$ and $\ell = 5$. (Since this paper was submitted for publication, we have proved the second conjecture [1], and two of us proved the third [5] when $k = 2$; part of the proof of the latter uses results of this paper, however, so this paper is not completely redundant.)

Conjecture 1.3 when $(k, \ell) = (2, 5)$ asserts that all pentagonal graphs have bounded chromatic number, where we say a graph is *pentagonal* if every induced odd cycle in it has length five (and in particular, it has no triangles). Pentagonal graphs might all be 4-colourable as far as we know (the 11-vertex Grötzsch graph is pentagonal and not 3-colourable), but at least they do indeed all have bounded chromatic number. The following is our main result:

1.4. *Every pentagonal graph is* 58 000*-colourable.*

The proof of 1.4 occupies almost the whole paper. (Much of the proof needs just that G is triangle-free and has no odd hole of length more than ℓ, for any fixed ℓ, and so we have written it in this generality wherever we could.) We prove: