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Quantum homomorphisms

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A B S T R A C T

A homomorphism from a graph X to a graph Y is an adjacency preserving map $f : V(X) \to V(Y)$. We consider a nonlocal game in which Alice and Bob are trying to convince a verifier with certainty that a graph X admits a homomorphism to Y. This is a generalization of the well-studied graph coloring game. Via systematic study of quantum homomorphisms we prove new results for graph coloring. Most importantly, we show that the Lovász theta number of the complement is a lower bound on the quantum chromatic number, the latter of which is not known to be computable. We also show that some of our newly introduced graph parameters, namely quantum independence and clique numbers, can differ from their classical counterparts while others, namely quantum odd girth, cannot. Finally, we show that quantum homomorphisms closely relate to zero-error channel capacity. In particular, we use quantum homomorphisms to construct graphs for which entanglement-assistance increases their one-shot zero-error capacity.

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1. Introduction

The c-coloring game on a graph X consists of two players, Alice and Bob, attempting to convince a referee that they have a c-coloring of X [13,8]. The game is played as follows: the referee sends each of the players a vertex of X, and each player responds with a color from $[c] = \{1, 2, \dots, c\}$. To win, the players must respond with the same color when they are sent the same vertex, and with different colors when they are sent adjacent vertices. The players are not allowed to communicate during the game but may agree on a strategy beforehand. It can be easily seen that deterministic players with shared randomness (classical players) can win the c-coloring game on X with certainty if and only if there exists a c-coloring of X. On the other hand, players allowed to make quantum measurements on some shared entangled state can sometimes win the c-coloring game on X with certainty even when X admits no c-coloring. Thus the quantum chromatic number, $\chi_q(X)$, is defined to be the smallest $c \in \mathbb{N}$ such that quantum players can win the c-coloring game on X [2].

Quantum strategies for the coloring game and the quantum chromatic number have been well-studied [2,7,12,24,21]. However, many questions remain unanswered. For example, it is not known whether $\chi_q(X)$ is computable, or whether there exists a family of graphs X_n such that $\lim_{n\to\infty} \chi_q(X_n) < \infty$ but $\lim_{n\to\infty} \chi(X_n) = \infty$. Furthermore, there are few lower bounds known for quantum chromatic number. The authors of [7] have shown that the orthogonal rank of a graph is a lower bound on a restricted version of quantum chromatic number, but the only general lower bound known is the clique number of a graph.

A *homomorphism* from a graph X to a graph Y is a function, ϕ, from the vertex set of X, denoted $V(X)$, to the vertex set of Y, denoted $V(Y)$, which preserves adjacency. More formally, $\phi : V(X) \to V(Y)$ is a homomorphism from X to Y if $\phi(x) \sim \phi(x')$ whenever $x \sim x'$, where "$\sim$" denotes adjacency. We will write $X \to Y$ if there exists a homomorphism from X to Y, and $X \nrightarrow Y$ if not. It is straightforward to see that a homomorphism from a graph X to the complete graph on c vertices, denoted K_c, is equivalent to a c-coloring of X. Thus homomorphisms are a natural generalization of colorings. There is a well-developed and beautiful theory around graph homomorphisms [17,16], and the study of them has given rise to original results on graph coloring.

Echoing the way in which homomorphisms generalize colorings, we have defined a homomorphism game which generalizes the coloring game. To play the (X, Y)-homomorphism game for graphs X and Y, each player is sent a vertex of X, and must respond with a vertex of Y. In order to win, the players must respond with the same vertex of Y when they are given the same vertex of X, and they must respond with adjacent vertices when given adjacent vertices. Similarly to the coloring game, classical players can win the (X, Y)-homomorphism game if and only if $X \to Y$. If quantum players can win the (X, Y)-homomorphism game, then we say that there is a *quantum homomorphism* from X to Y and write $X \xrightarrow{q} Y$.