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Notes

A circuit characterization of graphic matroids

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ABSTRACT

It is shown that a binary matroid is graphic if and only if it does not contain four circuits that interact is a particular way. This result generalizes a theorem of Little and Sanjith for planar graphs.

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1. Introduction

There are several known characterizations of graphic matroids. Generally speaking, these characterizations fall into two types: excluded-minor characterizations such as those given by Tutte [14,15] and Bixby [1], and characterizations using properties of cocircuits such as those given by Bixby [2], Bixby and Cunningham [3], Fournier [4], Lemos [7], Lemos, Reid, and Wu [8], Mighton [10], Sachs [12], Tutte [16], and Welsh [17]. Perhaps surprisingly, there does not seem to be a characterization of graphic matroids that is naturally expressed in terms of the circuits of a matroid. (The recent work of Geelen and Gerards [5] arguably falls in this category in that it characterizes, via a system of linear equations, when a set of fundamental circuits of a given binary matroid corresponds to that of a graphic matroid.)

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This paper presents a new characterization of those binary matroids that are graphic. In particular, it is shown that a binary matroid is graphic if and only if it does not contain four circuits that interact in a particular way. This result generalizes a characterization of planar graphs given by Little and Sanjith [9].

To state the characterization requires a couple of definitions. Let C and D be distinct circuits of a matroid M. A non-empty subset A of C is an *arc* of C *with respect to* D if $A \cup D$ contains at least two circuits, and A is minimal with respect to this property. (That is, $A \cup D$ is a *line* as defined by Tutte [15].) More generally, a non-empty subset A of C is an *arc* of C if there exists a circuit D such that A is an arc of C with respect to D. With respect to the arc A, C is the *primary* circuit and D is a *secondary* circuit.

Let A_1, A_2, and A_3 be distinct arcs of a circuit C of a matroid M. The set $\{A_1, A_2, A_3\}$ is called *incompatible* if $A_1 \cap A_2 \cap A_3 \neq \emptyset$ and no one of A_1, A_2, or A_3 is contained in the union of the other two.

The main theorem is the following.

Theorem 1. *A binary matroid is graphic if and only if it does not contain an incompatible set of arcs.* □

The next section contains the proof of Theorem 1. Any undefined matroid terminology is consistent with Oxley [11].

2. The proof

The first result of this section proves that no graphic matroid can contain an incompatible set of arcs, which is the easy direction of the proof of Theorem 1; this result is also in Little and Sanjith [9]. It is useful to observe the structure of arcs in graphic matroids. To this end, let G be a graph, and let $M(G)$ be the corresponding graphic matroid. Then, each circuit of $M(G)$ corresponds to the edge set of a cycle of G. Let C be a cycle of G, and let P be a subgraph of C such that the edge set of P is an arc of $E(C)$. Since P is a subgraph of C, either $P = C$ or P is the vertex-disjoint union of paths. Now, because arcs are minimal, in the latter case, it must be that P is a single path contained in C.

Lemma 2. *No graphic matroid contains an incompatible set of arcs.*

Proof. Suppose M is a graphic matroid that contains an incompatible set of arcs. Thus, there exists a graph G, a cycle C of G, and three paths P_1, P_2, and P_3 contained in C having an edge e in common, and such that there exists edges $e_1 \in P_1 - (P_2 \cup P_3)$, $e_2 \in P_2 - (P_1 \cup P_3)$, and $e_3 \in P_3 - (P_1 \cup P_2)$.

Since $e \in P_1 \cap P_2$, $P_1 \cup P_2$ is also a path properly contained in C. Moreover, observe that the edge e lies between e_1 and e_2 on $P_1 \cup P_2$. Since P_3 is a path contained in C and e_3 is not in $P_1 \cup P_2$, it follows that P_3 must also contain either e_1 or e_2, a contradiction. □