J. Combin. Theory Ser. B 119 (2016) 1–11

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series B

www.elsevier.com/locate/jctb

On Rota's Conjecture and nested separations in matroids ☆

Shalev Ben-David [a], Jim Geelen [b]

[a] *Computer Science and Artificial Intelligence Laboratory, MIT, Cambridge, MA, USA*
[b] *Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Canada*

A R T I C L E   I N F O

Article history:
Received 5 November 2013
Available online 23 December 2015

Keywords:
Matroids
Representation

A B S T R A C T

We prove that for each finite field $\mathbb{F}$ and integer $k \in \mathbb{Z}$ there exists $n \in \mathbb{Z}$ such that no excluded minor for the class of $\mathbb{F}$-representable matroids has n nested k-separations.

1. Introduction

We prove a partial result towards Rota's Conjecture [3].

Conjecture 1.1 *(Rota). For each finite field $\mathbb{F}$, there are, up to isomorphism, only finitely many excluded minors for the class of $\mathbb{F}$-representable matroids.*

A sequence $(A_1, B_1), \dots, (A_n, B_n)$ of k-separations in a matroid is said to be *nested* if $A_1 \subset A_2 \subset \cdots \subset A_n$. We prove the following theorem.

☆ This research was partially supported by an NSERC Discovery Grant (grant Nr. 203110-2006) held by Jim Geelen and by an NSERC Undergraduate Student Research Award held by Shalev Ben-David at the University of Waterloo in 2009.

http://dx.doi.org/10.1016/j.jctb.2015.11.002
0095-8956/

Theorem 1.2. *Let $\mathbb{F}$ be a finite field of order q, let k be a positive integer, and let $n = \mathrm{tower}(q, q, k, 6)$. Then no excluded minor for the class of $\mathbb{F}$-representable matroids admits a sequence of n nested k-separations.*

Here $\mathrm{tower}(a_1, a_2, \ldots, a_n) = a_1^{a_2^{\cdot^{\cdot^{\cdot^{a_n}}}}}$.

The special case of this result with $k = 3$ was proved by Oxley, Vertigan, and Whittle (personal communication).

We conclude the introduction with an application of Theorem 1.2 to branch-width; this corollary is proved in Section 6 where we also define branch-width.

Corollary 1.3. *For any finite field $\mathbb{F}$ of order q and positive integer k, if M is an excluded minor for the class of $\mathbb{F}$-representable matroids and M has branch-width k, then $|M| \leq \mathrm{tower}(3, q, q, k, 6)$.*

Corollary 1.3 improves the main result in [1] which gives a non-computable bound on $|M|$.

Our main result, Theorem 5.3, is an extension of Theorem 1.2 that involves representabilty over several fields.

2. Preliminaries

We use the following standard notation: we denote the power set of a set E by 2^E, and, if f is a function whose domain is a set E and $X \subseteq E$, then we denote $\{f(x) : x \in X\}$ by $f(X)$.

We follow the terminology of Oxley [2], except we write $|M|$ for the size of a matroid M. For a finite field $\mathbb{F}$, we define a *represented matroid* to be a pair (M, S) where $\mathrm{si}(S)$ is a projective geometry over $\mathbb{F}$ and M is a restriction of S. For a represented matroid (M, S) and $X \subseteq E(M)$, we denote $\mathrm{cl}_S(X)$ by $\mathrm{span}(X)$. For disjoint sets $D, C \subseteq E(M)$, we call $(M \setminus D/C, S/C)$ a *minor* of (M, S). For notational convenience we will usually refer to the represented matroid by M alone, and write S_M for S.

Let M be a matroid. For $X, Y \subseteq E(M)$, we define

$$\sqcap_M(X, Y) = r_M(X) + r_M(Y) - r_M(X \cup Y) \text{ and}$$
$$\lambda_M(X) = \sqcap_M(X, E(M) - X).$$

Thus, if M is representable, then

$$\sqcap_M(X, Y) = r_{S_M}(\mathrm{span}(X) \cap \mathrm{span}(Y)).$$

It is well-known that λ is submodular; that is,

$$\lambda_M(X) + \lambda_M(Y) \geq \lambda_M(X \cap Y) + \lambda_M(X \cup Y).$$