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On the maximum order of graphs embedded in surfaces

Eran Nevo [a], Guillermo Pineda-Villavicencio [b,a], David R. Wood [c]

[a] *Department of Mathematics, Ben-Gurion University of the Negev, Israel*
[b] *Centre for Informatics and Applied Optimisation, Federation University Australia, Australia*
[c] *School of Mathematical Sciences, Monash University, Australia*

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ABSTRACT

The maximum number of vertices in a graph of maximum degree $\Delta \geq 3$ and fixed diameter $k \geq 2$ is upper bounded by $(1+o(1))(\Delta-1)^k$. If we restrict our graphs to certain classes, better upper bounds are known. For instance, for the class of trees there is an upper bound of $(2+o(1))(\Delta-1)^{\lfloor k/2 \rfloor}$ for a fixed k. The main result of this paper is that graphs embedded in surfaces of bounded Euler genus g behave like trees, in the sense that, for large Δ, such graphs have orders bounded from above by

$$\begin{cases} c(g+1)(\Delta-1)^{\lfloor k/2 \rfloor} & \text{if } k \text{ is even} \\ c(g^{3/2}+1)(\Delta-1)^{\lfloor k/2 \rfloor} & \text{if } k \text{ is odd,} \end{cases}$$

where c is an absolute constant. This result represents a qualitative improvement over all previous results, even for planar graphs of odd diameter k. With respect to lower bounds, we construct graphs of Euler genus g, odd diameter k, and order $c(\sqrt{g}+1)(\Delta-1)^{\lfloor k/2 \rfloor}$ for some absolute constant $c > 0$. Our results answer in the negative a question of Miller and Širáň (2005).

E-mail addresses: nevoe@cs.bgu.ac.il (E. Nevo), work@guillermo.com.au (G. Pineda-Villavicencio), david.wood@monash.edu (D.R. Wood).

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1. Introduction

The degree–diameter problem asks for the maximum number of vertices in a graph of maximum degree $\Delta \geq 3$ and diameter $k \geq 2$. For general graphs the *Moore bound*,

$$\begin{aligned} M(\Delta, k) &:= 1 + \Delta + \Delta(\Delta-1) + \ldots + \Delta(\Delta-1)^{k-1} \\ &= (1+o(1))(\Delta-1)^k \quad \text{(for fixed } k\text{)}, \end{aligned}$$

provides an upper bound for the order of such a graph. The well-known de Bruijn graphs provide a lower bound of $\lfloor \Delta/2 \rfloor^k$ [2]. For background on this problem the reader is referred to the survey [13].

If we restrict our attention to particular graph classes, better upper bounds than the Moore bound are possible. For instance, a well-known result by Jordan [10] implies that every tree of maximum degree Δ and fixed diameter k has at most $(2+o(1))(\Delta-1)^{\lfloor k/2 \rfloor}$ vertices. For a graph class $\mathcal{C}$, we define $N(\Delta, k, \mathcal{C})$ to be the maximum order of a graph in $\mathcal{C}$ with maximum degree $\Delta \geq 3$ and diameter $k \geq 2$. We say $\mathcal{C}$ has *small order* if there exists a constant c and a function f such that $N(\Delta, k, \mathcal{C}) \leq c(\Delta-1)^{\lfloor k/2 \rfloor}$, for all $\Delta \geq f(k)$. The class of trees is a prototype class of small order.

For the class $\mathcal{P}$ of planar graphs, Hell and Seyffarth [9, Thm. 3.2] proved that $N(\Delta, 2, \mathcal{P}) = \lfloor \frac{3}{2}\Delta \rfloor + 1$ for $\Delta \geq 8$. Fellows et al. [6, Cor. 14] subsequently showed that $N(\Delta, k, \mathcal{P}) \leq ck\Delta^{\lfloor k/2 \rfloor}$ for every diameter k (see [7] for corresponding lower bounds). Notice that this does not prove that $\mathcal{P}$ has small order. Restricting $\mathcal{P}$ to even diameter assures small order, as shown by Tishchenko's upper bound of $(\frac{3}{2} + o(1))(\Delta-1)^{k/2}$, whenever $\Delta \in \Omega(k)$ [20, Thm. 1.1, Thm. 1.2]. Our first contribution is to prove that $N(\Delta, k, \mathcal{P}) \leq c(\Delta-1)^{\lfloor k/2 \rfloor}$ for $k \geq 2$ and $\Delta \in \Omega(k)$. That is, we show that the class of planar graphs has small order.

We now turn our attention to the class $\mathcal{G}_\Sigma$ of graphs embeddable in a surface[1] Σ of Euler genus g. For diameter 2 graphs, Knor and Širáň [11, Thm. 1, Thm. 2] showed that $N(\Delta, 2, \mathcal{G}_\Sigma) = N(\Delta, 2, \mathcal{P}) = \lfloor \frac{3}{2}\Delta \rfloor + 1$, provided $\Delta \in \Omega(g^2)$. Šiagiová and Simanjuntak [17, Thm. 1] proved for all diameters k the upper bound

$$N(\Delta, k, \mathcal{G}_\Sigma) \leq c(g+1)k(\Delta-1)^{\lfloor k/2 \rfloor}.$$

The main contribution of this paper, Theorem 1 below, is to show that the class of graphs embedded in a fixed surface Σ has small order.

Theorem 1. *There exists an absolute constant c such that, for every surface Σ of Euler genus g,*

[1] A *surface* is a compact (connected) 2-manifold (without boundary). Every surface is homeomorphic to the sphere with h handles or the sphere with c cross-caps [14, Thm. 3.1.3]. The sphere with h handles has *Euler genus* $g := 2h$, while the sphere with c cross-caps has *Euler genus* $g := c$. For a surface Σ and a graph G embedded in Σ, the (topologically) connected components of $\Sigma - G$ are called *faces*. A face homeomorphic to the open unit disc is called *2-cell*, and an embedding with only 2-cell faces is called a *2-cell embedding*. Every face in an embedding is bounded by a closed walk called a *facial walk*.