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Three-edge-colouring doublecross cubic graphs

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ABSTRACT

A graph is *apex* if there is a vertex whose deletion makes the graph planar, and *doublecross* if it can be drawn in the plane with only two crossings, both incident with the infinite region in the natural sense. In 1966, Tutte [9] conjectured that every two-edge-connected cubic graph with no Petersen graph minor is three-edge-colourable. With Neil Robertson, two of us showed that this is true in general if it is true for apex graphs and doublecross graphs [6,7]. In another paper [8], two of us solved the apex case, but the doublecross case remained open. Here we solve the doublecross case; that is, we prove that every two-edge-connected doublecross cubic graph is three-edge-colourable. The proof method is a variant on the proof of the four-colour theorem given in [5].

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1. Introduction

In this paper, all graphs are finite and simple. The four-colour theorem (4CT), that every planar graph is four-colourable, was proved by Appel and Haken [1,2] in 1977, and a simplified proof was given in 1997 by three of us, with Neil Robertson [5]. The 4CT is equivalent to the statement that every two-edge-connected cubic planar graph is three-edge-colourable, and a strengthening was proposed as a conjecture by Tutte in 1966 [9]; that every two-edge-connected cubic graph with no Petersen graph minor is three-edge-colourable. This paper is a step in the proof of Tutte's conjecture.

A graph G is *apex* if $G \setminus v$ is planar for some vertex v (we use $\setminus$ to denote deletion); and a graph G is *doublecross* if it can be drawn in the plane with only two crossings, both on the infinite region in the natural sense.

It is easy to check that apex and doublecross graphs do not contain the Petersen graph as a minor; but there is also a converse. Let us say a graph G is *theta-connected* if G is cubic and has girth at least five, and $|\delta_G(X)| \geq 6$ for all $X \subseteq V(G)$ with $|X|, |V(G) \setminus X| \geq 6$. ($\delta_G(X)$ denotes the set of edges of G with one end in X and one end in $V(G) \setminus X$.) Two of us (with Robertson) proved in [6] that every theta-connected graph with no Petersen graph minor is either apex or doublecross (with one exception, that is three-edge-colourable); and in [7] that every minimal counterexample to Tutte's conjecture was either apex or theta-connected. It follows that every minimal counterexample to Tutte's conjecture is either apex or doublecross, and so to prove the conjecture in general, it suffices to prove it for apex graphs and for doublecross graphs. Two of us proved in [8] that every two-edge-connected apex cubic graph is three-edge-colourable, so all that remains is the doublecross case, which is the objective of this paper. Our main theorem is:

1.1. *Every two-edge-connected doublecross cubic graph is three-edge-colourable.*

The proof method is by modifying the proof of the 4CT given in [5]. Again we give a list of reducible configurations (the definition of "reducible" has to be modified to accommodate the two pairs of crossing edges), and a discharging procedure to prove that one of these configurations must appear in every minimal counterexample (and indeed in every non-apex theta-connected doublecross graph). This will prove that there is no minimal counterexample, and so the theorem holds. Happily, the discharging rules given in [5] still work without any modification.

2. Crossings

We are only concerned with graphs that can be drawn in the plane with two crossings, and one might think that these are not much different from planar graphs, and perhaps one could just *use* the 4CT rather than going to all the trouble of repeating and modifying its proof. For graphs with one crossing this is true: here is a pretty theorem of Jaeger [4] (we include a proof because we think it is of interest):