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A characterisation of the generic rigidity of 2-dimensional point–line frameworks

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ABSTRACT

A 2-dimensional point–line framework is a collection of points and lines in the plane which are linked by pairwise constraints that fix some angles between pairs of lines and also some point–line and point–point distances. It is rigid if every continuous motion of the points and lines which preserves the constraints results in a point–line framework which can be obtained from the initial framework by a translation or a rotation. We characterise when a generic point–line framework is rigid. Our characterisation gives rise to a polynomial algorithm for solving this decision problem.

1. Introduction

A point–line framework is a collection of points and lines in d-dimensional Euclidean space which are linked by pairwise constraints that fix the angles between some pairs of lines, the distances between some pairs of points and the distances between some pairs of points and lines. The placing of the pairwise constraints is represented by a point–line graph where the vertices in the graph correspond to the points and lines, and an edge in

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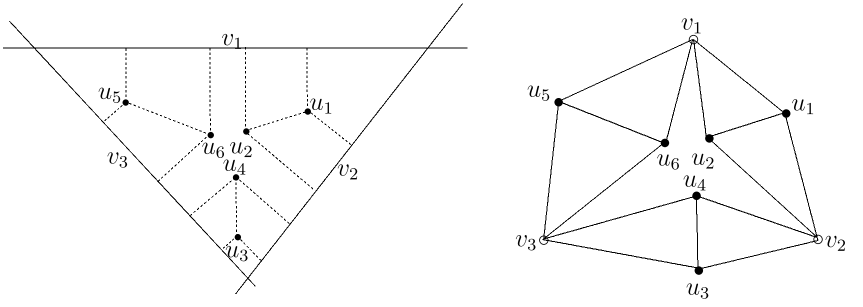

Fig. 1. A point–line framework in $\mathbb{R}^2$ and its associated point–line graph.

the graph corresponds to the existence of a pairwise constraint. A point–line framework is obtained from a point–line graph by assigning coordinates to the points and lines, see Fig. 1.

A point–line framework is rigid if every continuous motion of the points and lines which preserves the constraints results in a point–line framework which can be obtained from the initial framework by an isometry of the whole space.

The constraints of a point–line framework determine its rigidity matrix, in which rows are indexed by the edges of its point–line graph G. The rigidity matroid of the framework is the matroid on the edge set of G defined by the linear independence of the rows of the rigidity matrix. Since this linear independence will be the same for all generic frameworks with the same point–line graph, the rigidity matroid of a generic framework is completely determined by the dimension and the point–line graph. We will denote the 2-dimensional point–line rigidity matroid of a point–line graph G by $\mathcal{M}_{PL}(G)$ (or simply $\mathcal{M}_{PL}$ where the graph is implied).

Point–line frameworks with no lines correspond to the much studied bar-joint frameworks. Such frameworks provide a model for a variety of physical systems such as bar and joint structures [29] (where points correspond to universal joints and bars correspond to distance constraints) or molecular structures [14] (where points correspond to atoms and distance constraints correspond to bonds).

Laman [20] obtained the following characterisation of independence in the generic 2-dimensional bar-joint rigidity matroid. Given a graph $G = (V, E)$, let $\nu : 2^E \to \mathbb{Z}$ by taking $\nu(S)$ to be the number of vertices incident to S for all $S \subseteq E$. Then S is independent in the generic 2-dimensional bar-joint rigidity matroid of G if and only if $|S'| \leq 2\nu(S') - 3$ for all $\emptyset \neq S' \subseteq S$. The analogous condition that $|S'| \leq d\nu(S') - d(d+1)/2$ is a necessary condition for independence in the d-dimensional bar-joint rigidity matroid but it is not sufficient when $d \geq 3$. Characterising independence in the d-dimensional bar-joint rigidity matroid is an important open problem. We refer the reader to the survey article of Whiteley [29] for more information on bar-joint frameworks.

The point–line graph on the right of Fig. 1 can be used to illustrate the difference between point–line and bar-joint frameworks. We can use Laman's theorem to deduce that every generic realisation of this graph as a 2-dimensional bar-joint framework (with