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Maximum even factors of graphs

Fuyuan Chen [a], Genghua Fan [b,*,1]

[a] Institute of Statistics and Applied Mathematics, Anhui University of Finance and Economics, Bengbu, Anhui 233030, China

[b] Center for Discrete Mathematics, Fuzhou University, Fuzhou, Fujian 350108, China

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ABSTRACT

A spanning subgraph F of a graph G is called an even factor of G if each vertex of F has even degree at least 2 in F. Kouider and Favaron proved that if a graph G has an even factor, then it has an even factor F with $|E(F)| \geq \frac{9}{16}(|E(G)| + 1)$. In this paper we improve the coefficient $\frac{9}{16}$ to $\frac{4}{7}$, which is best possible. Furthermore, we characterize all the extremal graphs, showing that if $|E(H)| \leq \frac{4}{7}(|E(G)|+1)$ for every even factor H of G, then G belongs to a specified class of graphs.

1. Introduction

The graphs considered in this paper are finite, undirected, and with no loops or multiple edges. The sets of vertices and edges of a graph G are denoted by $V(G)$ and $E(G)$, respectively. Let H be a subgraph of G and $v \in V(G)$. The *degree* of v in H, denoted by $d_H(v)$, is the number of vertices in H which are joined to v. When $H = G$, we simply use $d(v)$ instead of $d_H(v)$. The *minimum degree* of G is denoted by

* Corresponding author.
E-mail addresses: chenfuyuan19871010@163.com (F. Chen), fan@fzu.edu.cn (G. Fan).

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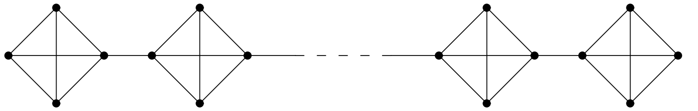

Fig. 1. Examples showing that the bound is best possible.

$\delta(G) = \min\{d(v) : v \in V(G)\}$. H is called a *spanning subgraph* of G if $V(H) = V(G)$. A *factor* of G is a spanning subgraph of G in which each vertex has positive degree. An *even graph* is one in which each vertex has even degree. Thus an even factor of G is a spanning even subgraph in which each vertex has positive even degree. A 2-*factor* of G is an even factor in which each vertex has degree 2.

It is well known that a bridgeless cubic graph G has a 2-factor F with $|E(F)| = \frac{2}{3}|E(G)|$. What can we say about non-cubic graphs? Clearly, the condition $\delta(G) \geq 2$ does not imply that G has a 2-factor. Fleischner [2] showed that if G is a bridgeless graph with minimum degree $\delta(G) \geq 3$, then it has an even factor. Lai and Chen [4] showed that such graphs have an even factor F with $|E(F)| \geq \frac{2}{3}|E(G)|$. Replacing $\delta(G) \geq 3$ by "having an even factor" and dropping the requirement of bridgeless, Favaron and Kouider [1] showed that if a graph G has an even factor, then it has an even factor F with $|E(F)| \geq \frac{9}{16}(|E(G)| + 1)$. In the same paper [1], they gave the following examples illustrated in Fig. 1.

The graph G consists of m disjoint copies of K_4 (complete graph on 4 vertices) and a set of $m-1$ edges, in which $\delta(G) \geq 3$. It is easy to see that G has an even factor, but no even factor contains edges more than $\frac{4}{7}(|E(G)| + 1)$. With these examples, Favaron and Kouider [1] were not sure that the coefficient $\frac{9}{16}$ they obtained is best possible and asked whether the best possible coefficient $\frac{4}{7}$ can be achieved. In this paper, we show that $\frac{4}{7}$ is achievable and characterize all the extremal graphs that achieve the coefficient $\frac{4}{7}$.

For an integer $m \geq 1$, let T_m be the graph obtained from m disjoint copies of K_4 by adding $m - 1$ edges such that contracting every K_4 into a single vertex results in a tree of $m - 1$ edges. The graphs in Fig. 1 are special cases in which the resulting tree is a path of length $m - 1$. The following theorem is proved in the next section.

Theorem 1.1. *If a graph G has an even factor, then it has an even factor F with $|E(F)| \geq \frac{4}{7}(|E(G)| + 1)$. Moreover, if $|E(H)| \leq \frac{4}{7}(|E(G)| + 1)$ for every even factor H of G, then $G = T_m$, for some $m \geq 1$.*

Another problem related to large even factors discussed in this paper is the one about the existence of large connected even factors. Catlin (see [5]) conjectured that if a graph G has a connected even factor, then it has a connected even factor F with $|E(F)| \geq \frac{2}{3}|E(G)|$. Li et al. [5] disproved the conjecture by constructing a class of graphs G that have connected even factors, but no connected even factor F with $|E(F)| \geq \frac{2}{3}|E(G)|$, and conjectured $\frac{3}{5}$ is the right coefficient for connected even factors in graphs that have connected even factors. We point out that the graphs in Fig. 1 do not have connected even factors. Having a connected even factor is a quite strong property. Determining whether