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Möbius conjugation and convolution formulae ☆

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ABSTRACT

Let P be a locally finite poset with the interval space $\mathrm{Int}(P)$, and R be a ring with identity. We shall introduce the Möbius conjugation μ^* sending each function $f : P \to R$ to an incidence function $\mu^*(f) : \mathrm{Int}(P) \to R$ such that $\mu^*(fg) = \mu^*(f) * \mu^*(g)$. Taking P to be the intersection poset of a hyperplane arrangement $\mathcal{A}$, we shall obtain a convolution identity for the number $r(\mathcal{A})$ of regions and the number $b(\mathcal{A})$ of relatively bounded regions, and a reciprocity theorem of the characteristic polynomial $\chi(\mathcal{A}, t)$ which gives a combinatorial interpretation of the values $|\chi(\mathcal{A}, -q)|$ for large primes q. Moreover, all known convolution identities on Tutte polynomials of matroids will be direct consequences after specializing the poset P and functions f, g.

1. Möbius conjugation

We use the definitions and notations of posets from [12], and all posets in this paper are assumed to be locally finite. Let P be a poset and R a ring with identity. Denote by $\mathrm{Int}(P)$ the interval space of P and $\mathcal{I}(P, R) = \{\alpha : \mathrm{Int}(P) \to R\}$ the incidence algebra of P whose multiplication structure is given by the convolution product, i.e., for any $\alpha, \beta \in \mathcal{I}(P, R)$ and $x \leq y$ in P,

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$$[\alpha * \beta](x,y) = \sum_{x \le z \le y} \alpha(x,z)\,\beta(z,y), \quad \forall\, \alpha, \beta \in \mathcal{I}(P,R).$$

Let R^P be the ring of all functions $f : P \to R$ whose ring structure is given by point-wise multiplication and addition. Define the *Möbius conjugation* $\mu^* : R^P \to \mathcal{I}(P,R)$ to be

$$\mu^*(f) = \mu * \delta(f) * \zeta, \quad \forall\, f \in R^P,$$

where ζ is the constant function $\zeta(x,y) = 1$ for $x \le y$ in P, μ is the Möbius function of P, i.e., the convolution inverse of ζ, and the map $\delta : R^P \to \mathcal{I}(P,R)$ is defined by $\delta(f)(x,y) = f(x)$ if $x = y$ and 0 otherwise, for all $f \in R^P$ and $x \le y$ in P.

Theorem 1.1. *With above settings, the map μ^* is a ring monomorphism, i.e.,*

$$\mu^*(fg) = \mu^*(f) * \mu^*(g), \quad \forall\, f, g \in R^P.$$

Proof. Given $f, g \in R^P$, it is obvious that $\delta(fg) = \delta(f) * \delta(g)$, then

$$\mu^*(fg) = \mu * \delta(f) * \zeta * \mu * \delta(g) * \zeta = \mu^*(f) * \mu^*(g).$$

So μ^* is a homomorphism of rings. To prove the injectivity, suppose $\mu^*(f) = \mu^*(g)$ for some $f, g \in R^P$. Multiplying ζ on the left-hand side and μ on the right-hand side respectively, we obtain that $\delta(f) = \delta(g)$. Thus $f = g$. □

2. Convolution formula on characteristic polynomials

A hyperplane arrangement $\mathcal{A}$ in a vector space V is a finite collection of hyperplanes of V. The intersection semi-lattice $L(\mathcal{A})$ of $\mathcal{A}$ is defined to be the collection of all nonempty intersections of hyperplanes in $\mathcal{A}$, whose partial order is given by the inverse of set inclusion. Namely,

$$L(\mathcal{A}) = \{\cap_{H \in \mathcal{B}} H \mid \mathcal{B} \subseteq \mathcal{A}\},$$

whose minimal element is $\hat{0} = \cap_{H \in \emptyset} H := V \in L(\mathcal{A})$. Artificially adding a maximal element $\hat{1} = \emptyset$ to $L(\mathcal{A})$, $L(\mathcal{A})$ then becomes a geometric lattice, denoted $L^*(\mathcal{A}) = L(\mathcal{A}) \cup \{\hat{1}\}$ and called the *reduced intersection lattice* of $\mathcal{A}$. With the assumptions $\dim(\hat{1}) = \infty$ and $t^\infty = 0$, the *characteristic polynomial* $\chi(\mathcal{A}, t) \in \mathbb{C}[t]$ of $\mathcal{A}$ can be written as

$$\chi(\mathcal{A}, t) := \sum_{X \in L(\mathcal{A})} \mu(\hat{0}, X)\, t^{\dim(X)} = \sum_{X \in L^*(\mathcal{A})} \mu(\hat{0}, X)\, t^{\dim(X)}.$$