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On the chromatic number of general Kneser hypergraphs ☆

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ABSTRACT

In a break-through paper, Lovász [20] determined the chromatic number of Kneser graphs. This was improved by Schrijver [27], by introducing the Schrijver subgraphs of Kneser graphs and showing that their chromatic number is the same as that of Kneser graphs. Alon, Frankl, and Lovász [2] extended Lovász's result to the usual Kneser hypergraphs and one of our main results is to extend this to a new family of general Kneser hypergraphs. Moreover, as a special case, we settle a question from Naserasr and Tardif [26].

In 2011, Meunier introduced almost 2-stable Kneser hypergraphs and determined their chromatic number as an approach to a supposition of Ziegler [35] and a conjecture of Alon, Drewnowski, and Łuczak [3]. In this work, our second main result is to improve this by computing the chromatic number of a large family of Schrijver hypergraphs. Our last main result is to prove the existence of a completely multicolored complete bipartite graph in every coloring of a graph which extends a result of Simonyi and Tardos [29].

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The first two results are proved using a new improvement of the Dol'nikov–Kříž [7,18] bound on the chromatic number of general Kneser hypergraphs.

1. Introduction

In this work, we derive a substantial improvement on the Dol'nikov–Kříž lower bound for the chromatic number of general Kneser hypergraphs (see Theorem A). This improvement is used in our determination of the chromatic number of some families of hypergraphs. The problem of finding a lower bound for the chromatic number of hypergraphs has been studied in the literature, see [7,18,19,28,29,35,36]. As in [24,35], our proofs rely on the Z_p-Tucker lemma, which is a combinatorial generalization of the Borsuk–Ulam theorem. This lemma has also been used to simplify the evaluation of the chromatic number for some families of hypergraphs, e.g., almost 2-stable Kneser hypergraphs. Like the Borsuk–Ulam theorem, the Z_p-Tucker lemma is used to obtain lower bounds for the chromatic number of hypergraphs, see [24,35].

First, in this section, we setup some notation and terminology. Hereafter, the symbol $[n]$ stands for the set $\{1,\ldots,n\}$. A *hypergraph* $\mathcal{F}$ is an ordered pair $(V(\mathcal{F}),E(\mathcal{F}))$, where $V(\mathcal{F})$ (the *vertex set*) is a finite set and $E(\mathcal{F})$ (the *hyperedge set*) is a family of distinct nonempty subsets of $V(\mathcal{F})$. If every hyperedge of a hypergraph has size r, then it is called an *r-uniform* hypergraph. A *t-coloring* of a hypergraph $\mathcal{F}$ is a mapping $h : V(\mathcal{F}) \longrightarrow [t] = \{1,2,\ldots,t\}$ such that no hyperedge is monochromatic. The minimum t such that there exists a t-coloring of the hypergraph $\mathcal{F}$ is called its *chromatic number*, and is denoted by $\chi(\mathcal{F})$. If $\mathcal{F}$ has a hyperedge of size 1, then we define the chromatic number of $\mathcal{F}$ to be infinite. Throughout this paper, for any hypergraph $\mathcal{F}$ with the vertex set $V(\mathcal{F}) = \{v_1,v_2,\ldots,v_n\}$, we consider a fixed bijective labeling $L_{\mathcal{F}} : [n] \longrightarrow V(\mathcal{F})$, i.e., a bijective map from $\{1,2,\ldots,n\}$ to $V(\mathcal{F})$. By abuse of notation, we assume that i and $L_{\mathcal{F}}(i)$ are referring to the same vertex of $\mathcal{F}$ and we use these representations interchangeably. The labeling $L_{\mathcal{F}}$ allows us to identify the vertex set of $\mathcal{F}$ with the set $[n]$. Let t be a positive integer and $N = (N_1,N_2,\ldots,N_t)$, where the N_i's are pairwise disjoint subsets of $V = [n]$. The *induced hypergraph* $\mathcal{F}_{|_N}$ has $\bigcup_{i=1}^{t} N_i$ and $\{A \in E(\mathcal{F}) : \exists i;\ 1 \leq i \leq t, A \subseteq N_i\}$ as vertex set and hyperedge set, respectively. For any hypergraph $\mathcal{F} = (V(\mathcal{F}),E(\mathcal{F}))$ and positive integer $r \geq 2$, the *general Kneser hypergraph* $\mathrm{KG}^r(\mathcal{F})$ is an r-uniform hypergraph whose vertex set is $E(\mathcal{F})$ and whose hyperedge set consists of all r-tuples of pairwise disjoint hyperedges of $\mathcal{F}$. In this terminology, the usual Kneser hypergraph is the hypergraph $\mathrm{KG}^r(\binom{[n]}{k})$, where $\binom{[n]}{k}$ denotes the hypergraph with vertex set $[n]$ and hyperedge set containing all k-subsets of $[n]$. A hypergraph $\mathcal{F}$ provides a *Kneser representation* for a graph G, if G and $\mathrm{KG}^2(\mathcal{F})$ are isomorphic. It is known that every graph has various Kneser representations; see [14] for more details.