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Coloring digraphs with forbidden cycles

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ABSTRACT

Let k and r be two integers with $k \geq 2$ and $k \geq r \geq 1$. In this paper we show that (1) if a strongly connected digraph D contains no directed cycle of length 1 modulo k, then D is k-colorable; and (2) if a digraph D contains no directed cycle of length r modulo k, then D can be vertex-colored with k colors so that each color class induces an acyclic subdigraph in D. Our results give affirmative answers to two questions posed by Tuza in 1992. Moreover, the second one implies the following strong form of a conjecture of Diwan, Kenkre and Vishwanathan: If an undirected graph G contains no cycle of length r modulo k, then G is k-colorable if $r \neq 2$ and $(k+1)$-colorable otherwise. Our results also strengthen several classical theorems on graph coloring proved by Bondy, Erdős and Hajnal, Gallai and Roy, Gyárfás, etc.

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1. Introduction

Digraphs considered in this paper contain neither loops nor parallel arcs. By a *cycle* (respectively *path*) in a digraph we mean a simple and directed one throughout. Let D be a digraph. As usual, the *underlying graph* of D, denoted by G, is obtained from D by replacing each arc with an edge having the same ends. A *proper k-coloring* of D is simply a proper k-coloring of G. Thus D is k-colorable if G is k-colorable, and the *chromatic number* $\chi(D)$ of D is exactly $\chi(G)$. An *acyclic k-coloring* of D is an assignment of k colors to the vertices of D so that each color class induces an acyclic subdigraph in D. The *dichromatic number* $\chi_A(D)$ of D is the minimum k for which D admits an acyclic k-coloring. Clearly, $\chi_A(D) \le \chi(D)$; this inequality, however, need not hold equality in general.

Classical digraph coloring arises in a wide variety of applications, and hence it has attracted many research efforts. As it is *NP*-hard to determine the chromatic number of a given digraph, the focus of extensive research has been on good bounds. A fundamental theorem due to Gallai and Roy [10,22] asserts that the chromatic number of a digraph is bounded above by the number of vertices in a longest path. It is natural to further explore the connection between chromatic number and cycle lengths. To get meaningful results in this direction, a common practice is to impose strong connectedness on digraphs we consider. Bondy [3] showed that the chromatic number of a strongly connected digraph D is at most its *circumference*, the length of a longest cycle in D. In [23], Tuza proved that if an undirected graph G contains no cycle whose length minus one is a multiple of k, then G is k-colorable. He also asked whether or not similar results can be obtained for digraphs in terms of cycle lengths that belong to prescribed residue classes. One objective of our paper is to give an affirmative answer to this question, which strengthens, among others, all the theorems stated above.

Theorem 1. *Let $k \ge 2$ be an integer. If a strongly connected digraph D contains no directed cycle of length 1 modulo k, then $\chi(D) \le k$.*

We point out that the bound is sharp for infinitely many digraphs, such as strongly connected tournaments with an even number of vertices.

The *odd circumference* of a graph G (directed or undirected), denoted by $l(G)$, is the length of a longest odd cycle (if any) in G. We set $l(G) = 1$ if G contains no odd cycle. A corollary of the above theorem is the following statement, which is of interest in its own right and is in the same spirit as the aforementioned Bondy's theorem [3].

Theorem 2. *For every strongly connected digraph D, we have $\chi(D) \le l(D) + 1$.*

It was shown by Erdős and Hajnal [9] that $\chi(G) \le l(G) + 1$ for any undirected graph G; the bound is achieved only when G contains a complete subgraph with $l(G)+1$ vertices (see Kenkre and Vishwanathan [17]). So a natural question to ask is whether