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Maximizing proper colorings on graphs ☆

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A B S T R A C T

The number of proper q-colorings of a graph G, denoted by $P_G(q)$, is an important graph parameter that plays fundamental role in graph theory, computational complexity theory and other related fields. We study an old problem of Linial and Wilf to find the graphs with n vertices and m edges which maximize this parameter. This problem has attracted much research interest in recent years, however little is known for general m, n, q. Using analytic and combinatorial methods, we characterize the asymptotic structure of extremal graphs for fixed edge density and q. Moreover, we disprove a conjecture of Lazebnik, which states that the Turán graph $T_s(n)$ has more q-colorings than any other graph with the same number of vertices and edges. Indeed, we show that there are infinite many counterexamples in the range $q = O(s^2/\log s)$. On the other hand, when q is larger than some constant times $s^2/\log s$, we confirm that the Turán graph $T_s(n)$ asymptotically is the extremal graph achieving the maximum number of q-colorings. Furthermore, other (new and old) results on various instances of the Linial–Wilf problem are also established.

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1. Introduction

A *proper q-coloring* of a graph G is an assignment mapping every vertex to one of the q colors in such a way that no two adjacent vertices receive the same color. Let $P_G(q)$ denote the number of proper q-colorings in a graph G. Introduced by Birkhoff [2] in 1912, who proved that $P_G(q)$ is always a polynomial in q, this important graph parameter, as now commonly referred to as *chromatic polynomial* of G, has been extensively investigated over the past century. As it is already NP-hard to determine whether this number $P_G(q)$ is nonzero (even for $q = 3$ and planar graph G), the focus of substantial research has been to obtain good bounds for $P_G(q)$ over various families of graphs.

The original motive for Birkhoff [2] to consider the chromatic polynomial was the famous four-color conjecture (now a theorem), which equivalently asserts that the minimum $P_G(4)$ over all planar graphs is at least one. For every $q \geq 5$, it was obtained by Birkhoff in [3] that $P_G(q) \geq q(q-1)(q-2)(q-3)^{n-3}$ for every planar graph G with n vertices, which is also sharp. Motivated from computational complexity, Linial [11] arrived at the problem of minimizing the number of *acyclic orientations* of graph G, which equals $|P_G(-1)|$ by a result of Stanley [18], over the family $\mathcal{F}_{n,m}$ of graphs with n vertices and m edges. He gave a surprising answer that for any n, m, there exists a universal graph minimizing $|P_G(q)|$ over the family $\mathcal{F}_{n,m}$ for *every* integer q. This graph is obtained from a clique K_k by adding $n - k - 1$ isolated vertices and an extra vertex adjacent to l vertices of the clique K_k, where $k > l \geq 0$ are the unique integers satisfying $\binom{k}{2} + l = m$.

Linial [11] then asked for the counterpart of his result, that is, to maximize $|P_G(q)|$ over all graphs with n vertices and m edges for integers q. Wilf (see [1,20]) independently raised the same maximization problem from a different point of view, the *backtracking* algorithm for finding a proper q-coloring. Since then, this problem has been the subject of extensive research, and many upper bounds on $P_G(q)$ over the family $\mathcal{F}_{n,m}$ have been obtained (see, for instance, [5–7,12,15]). The case $q = 2$ (for all n, m) was solved by Lazebnik in [6] completely. In the same paper, Lazebnik conjectured that in the range $m \leq n^2/4$, the graphs with n vertices and m edges maximizing the number of 3-colorings must be complete bipartite graphs $K_{a,b}$ minus the edges of some star, plus isolated vertices. This was confirmed in a breakthrough paper [13] of Loh, Pikhurko and Sudakov, who further determined the asymptotic values of a, b. For $q \geq 4$, they also showed that the same graphs achieve the maximum number of q-colorings, for all sufficiently large $m < \kappa_q n^2$ where $\kappa_q \approx 1/(q \log q)$. In fact, Loh et al. [13] provided a general approach which enables to find the asymptotic solution of the Linial–Wilf problem by reducing it to a quadratically constrained linear problem, which we shall introduce in Section 2. Despite the efforts by various researchers, still very little was known for general m, n, q. “Perhaps part of the difficulty for general m, n, q stems from the fact that the maximal graphs are substantially more complicated than the minimal graphs that Linial found” (quoted from [13]).