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Edge-colouring seven-regular planar graphs

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ABSTRACT

A conjecture due to the fourth author states that every d-regular planar multigraph can be d-edge-coloured, provided that for every odd set X of vertices, there are at least d edges between X and its complement. For $d = 3$ this is the four-colour theorem, and the conjecture has been proved for all $d \leq 8$, by various authors. In particular, two of us proved it when $d = 7$; and then three of us proved it when $d = 8$. The methods used for the latter give a proof in the $d = 7$ case that is simpler than the original, and we present it here.

1. Introduction

Let G be a graph. (Graphs in this paper are finite, and may have loops or parallel edges.) If $X \subseteq V(G)$, $\delta_G(X) = \delta(X)$ denotes the set of all edges of G with an end in X and an end in $V(G) \setminus X$. We say that G is *oddly d-edge-connected* if $|\delta(X)| \geq d$ for all

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odd subsets X of $V(G)$. The following conjecture [8] was proposed by the fourth author in about 1973.

1.1. Conjecture. *If G is a d-regular planar graph, then G is d-edge-colourable if and only if G is oddly d-edge-connected.*

The "only if" part is true, and some special cases of the "if" part of this conjecture have been proved.

- For $d = 3$ it is the four-colour theorem, and was proved by Appel and Haken [1,2,7];
- for $d = 4, 5$ it was proved by Guenin [6];
- for $d = 6$ it was proved by Dvorak, Kawarabayashi and Kral [4];
- for $d = 7$ it was proved by the second and third authors and appears in the Master's thesis [5] of the former;
- for $d = 8$ it was proved by three of us [3].

The methods of [3] can be adapted to yield a proof of the result for $d = 7$, that is shorter and simpler than that of [5]. Since in any case the original proof appears only in a thesis, we give the new one here. Thus, we show

1.2. *Every 7-regular oddly 7-edge-connected planar graph is 7-edge-colourable.*

All these proofs (for $d > 3$), including ours, assume the truth of the result for $d - 1$. Thus we need to assume the truth of the result for $d = 6$. Some things that are proved in [3] are true for all d, and we sometimes cite results from that paper.

2. An unavoidable list of reducible configurations

Any 7-regular planar graph has parallel edges, and it is helpful to reformulate the problem in terms of the underlying simple graph; then we have a number for each edge, recording the number of parallel edges it represents. Let us say a *d-target* is a pair (G, m) with the following properties (where for $F \subseteq E(G)$, $m(F)$ denotes $\sum_{e \in F} m(e)$):

- G is a simple graph drawn in the plane;
- $m(e) \geq 0$ is an integer for each edge e;
- $m(\delta(v)) = d$ for every vertex v; and
- $m(\delta(X)) \geq d$ for every odd subset $X \subseteq V(G)$.

In this language, 1.1 says that for every d-target (G, m), there is a list of d perfect matchings of G such that every edge e of G is in exactly $m(e)$ of them. (The elements of a list need not be distinct.) If there is such a list we call it a *d-edge-colouring*, and say that (G, m) is *d-edge-colourable*. For an edge $e \in E(G)$, we call $m(e)$ the *multiplicity*