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Edge-colouring eight-regular planar graphs

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ABSTRACT

It was conjectured by the third author in about 1973 that every d-regular planar graph (possibly with parallel edges) can be d-edge-coloured, provided that for every odd set X of vertices, there are at least d edges between X and its complement. For $d = 3$ this is the four-colour theorem, and the conjecture has been proved for all $d \leq 7$, by various authors. Here we prove it for $d = 8$.

1. Introduction

One form of the four-colour theorem, due to Tait [9], asserts that a 3-regular planar graph can be 3-edge-coloured if and only if it has no cut-edge. But when can d-regular planar graphs be d-edge-coloured?

Let G be a graph. (Graphs in this paper are finite, and may have loops or parallel edges.) If $X \subseteq V(G)$, $\delta_G(X) = \delta(X)$ denotes the set of all edges of G with an end in X and an end in $V(G) \setminus X$. We say that G is *oddly d-edge-connected* if $|\delta(X)| \geq d$ for all odd subsets X of $V(G)$. Since every perfect matching contains an edge of $\delta(X)$ for every odd set $X \subseteq V(G)$, it follows that every d-regular d-edge-colourable graph is oddly

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d-edge-connected. (Note that for a 3-regular graph, being oddly 3-edge-connected is the same as having no cut-edge, because if $X \subseteq V(G)$, then $|\delta(X)| = 1$ if and only if $|X|$ is odd and $|\delta(X)| < 3$.) The converse is false, even for $d = 3$ (the Petersen graph is a counterexample); but for planar graphs perhaps the converse is true. That is the content of the following conjecture [8], proposed by the third author in about 1973.

1.1. Conjecture. *If G is a d-regular planar graph, then G is d-edge-colourable if and only if G is oddly d-edge-connected.*

Some special cases of this conjecture have been proved.

- For $d = 3$ it is the four-colour theorem, and was proved by Appel and Haken [1,2,7];
- for $d = 4, 5$ it was proved by Guenin [6];
- for $d = 6$ it was proved by Dvorak, Kawarabayashi and Kral [4];
- for $d = 7$ it was proved by Kawarabayashi and the second author, and appears in the Master's thesis [5] of the latter. The methods of the present paper can also be applied to the $d = 7$ case, resulting in a proof somewhat simpler than the original, and this simplified proof for the $d = 7$ case will be presented in another, four-author paper [3].

Here we prove the next case, namely:

1.2. *Every 8-regular oddly 8-edge-connected planar graph is 8-edge-colourable.*

All these proofs (for $d > 3$), including ours, proceed by induction on d. Thus we need to assume the truth of the result for $d = 7$.

2. An unavoidable list of reducible configurations

The graph we wish to edge-colour has parallel edges, but it is more convenient to work with the underlying simple graph. If H is d-regular and oddly d-edge-connected, then H has no loops, because for every vertex v, v has degree d, and yet $|\delta_H(v)| \geq d$. (We write $\delta(v)$ for $\delta(\{v\})$.) Thus to recover H from the underlying simple graph G say, we just need to know the number $m(e)$ of parallel edges of H that correspond to each edge e of G. Let us say a *d-target* is a pair (G, m) with the following properties (where for $F \subseteq E(G)$, $m(F)$ denotes $\sum_{e \in F} m(e)$):

- G is a simple graph drawn in the plane;
- $m(e) \geq 0$ is an integer for each edge e;
- $m(\delta(v)) = d$ for every vertex v; and
- $m(\delta(X)) \geq d$ for every odd subset $X \subseteq V(G)$.