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Diameter critical graphs

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A B S T R A C T

A graph is called diameter-k-critical if its diameter is k, and the removal of any edge strictly increases the diameter. In this paper, we prove several results related to a conjecture often attributed to Murty and Simon, regarding the maximum number of edges that any diameter-k-critical graph can have. In particular, we disprove a longstanding conjecture of Caccetta and Häggkvist (that in every diameter-2-critical graph, the average edge-degree is at most the number of vertices), which promised to completely solve the extremal problem for diameter-2-critical graphs.

On the other hand, we prove that the same claim holds for all higher diameters, and is asymptotically tight, resolving the average edge-degree question in all cases except diameter-2. We also apply our techniques to prove several bounds for the original extremal question, including the correct asymptotic bound for diameter-k-critical graphs, and an upper bound of $(\frac{1}{6}+o(1))n^2$ for the number of edges in a diameter-3-critical graph.

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1. Introduction

An (x, y)*-path* is a path with endpoints x and y, and its *length* is its number of edges. We denote by $d_G(x, y)$ the smallest length of an (x, y)-path in a graph G, where we often drop the subscript if the graph G is clear from context. The *diameter* of G is the maximum of $d_G(x, y)$ over all pairs $\{x, y\}$. A graph G is said to be *diameter-critical* if for every edge $e \in G$, the deletion of e produces a graph $G - e$ with higher diameter.

The area of diameter-criticality is one of the oldest subjects of study in extremal graph theory, starting from papers of Erdős–Rényi [10], Erdős–Rényi–Sós [11], Murty–Vijayan [20], Murty [17–19], and Ore [21] from the 1960's. Many problems in this domain were investigated, such as that of minimizing the number of edges subject to diameter and maximum-degree conditions (see, e.g., Erdős–Rényi [10], Bollobás [1,2], Bollobás–Eldridge [3], Bollobás–Erdős [4]), controlling post-deletion diameter (Chung [8]), and vertex-criticality (Caccetta [5], Erdős–Howorka [9], Huang–Yeo [15], Chen–Füredi [7]), to name just a few.

The natural extremal problem of maximizing the number of edges (or equivalently, the average degree) in a diameter-critical graph also received substantial attention. Our work is inspired by the following long-standing conjecture of Ore [21], Plesník [22], Murty and Simon (see in [6]). A *diameter-k-critical* graph is a diameter-critical graph of diameter k.

Conjecture 1.1. *For each n, the unique diameter-2-critical graph which maximizes the number of edges is the complete bipartite graph $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$.*

Successively stronger estimates were proved in the 1980's by Plesník [22], Cacetta–Häggkvist [6], and Fan [12], culminating in a breakthrough by Füredi [13], who used a clever application of the Ruzsa–Szemerédi (6,3) theorem to prove the exact (non-asymptotic) result for large n. As current quantitative bounds on the (6,3) theorem are of tower-type, the constraint on n is quite intense, and there is interest in finding an approach which is free of Regularity-type ingredients. For example, the recent survey by Haynes, Henning, van der Merwe, and Yeo [14] discusses recent work following a different approach based upon total domination, but we do not pursue that direction in this paper.

One hope for a regularity-free method was proposed at around the origin of the early investigation. In their original 1979 paper, Caccetta and Häggkvist posed a very elegant stronger conjecture for a related problem, which would establish the extremal number of edges in diameter-2-critical graphs for all n. For an edge e, let its *edge-degree* $d(e)$ be the sum of the degrees of its endpoints, and let $\overline{d(e)}$ be the average edge-degree over all edges, so that in terms of the total number of edges m,

$$\overline{d(e)} = \frac{1}{m} \sum_{uv \in E(G)} (d_u + d_v) = \frac{1}{m} \sum_{v \in V(G)} d_v^2.$$