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Generic global rigidity of body-hinge frameworks

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ABSTRACT

A d-dimensional body-hinge framework is a structure consisting of rigid bodies in d-space in which some pairs of bodies are connected by a hinge, restricting the relative position of the corresponding bodies. The framework is said to be globally rigid if every other arrangement of the bodies and their hinges can be obtained by a congruence of the space. The combinatorial structure of a body-hinge framework can be encoded by a multigraph H, in which the vertices correspond to the bodies and the edges correspond to the hinges. We prove that a generic body-hinge realization of a multigraph H is globally rigid in $\mathbb{R}^d$, $d \geq 3$, if and only if $(\binom{d+1}{2}-1)H-e$ contains $\binom{d+1}{2}$ edge-disjoint spanning trees for all edges e of $(\binom{d+1}{2}-1)H$. (For a multigraph H and integer k we use kH to denote the multigraph obtained from H by replacing each edge e of H by k parallel copies of e.) This implies an affirmative answer to a conjecture of Connelly, Whiteley, and the first author.

We also consider bar-joint frameworks and show, for each $d \geq 3$, an infinite family of graphs satisfying Hendrickson's well-known necessary conditions for generic global rigidity in $\mathbb{R}^d$ (that is, $(d+1)$-connectivity and redundant rigidity) which are not generically globally rigid in $\mathbb{R}^d$. The existence of these

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families disproves a number of conjectures, due to Connelly, Connelly and Whiteley, and the third author, respectively.

1. Introduction

A d-dimensional (bar-and-joint) *framework* is a pair (G, p), where $G = (V, E)$ is a graph and p is a map from V to $\mathbb{R}^d$. We consider the framework to be a straight line *realization* of G in $\mathbb{R}^d$. Two realizations (G, p) and (G, q) of G are *equivalent* if $||p(u) - p(v)|| = ||q(u) - q(v)||$ holds for all pairs u, v with $uv \in E$, where $||.||$ denotes the Euclidean norm in $\mathbb{R}^d$. Frameworks (G, p), (G, q) are *congruent* if $||p(u) - p(v)|| = ||q(u) - q(v)||$ holds for all pairs u, v with $u, v \in V$.

We say that (G, p) is *globally rigid* in $\mathbb{R}^d$ if every d-dimensional realization of G which is equivalent to (G, p) is congruent to (G, p). The framework (G, p) is *rigid* if there exists an $\epsilon > 0$ such that, if (G, q) is equivalent to (G, p) and $||p(u) - q(u)|| < \epsilon$ for all $v \in V$, then (G, q) is congruent to (G, p). Intuitively, this means that if we think of a d-dimensional framework (G, p) as a collection of bars and joints where points correspond to joints and each edge to a rigid (i.e. fixed length) bar joining its end-points, then the framework is globally rigid if its bar lengths determine the realization up to congruence. It is rigid if every continuous motion of the joints that preserves all bar lengths must preserve all pairwise distances between the joints, see e.g. [27]. It is a hard problem to decide if a given framework is rigid or globally rigid. Indeed Saxe [17] showed that it is NP-hard to decide if even a 1-dimensional framework is globally rigid and Abbot [1] showed that the rigidity problem is NP-hard for 2-dimensional frameworks. These problems become more tractable, however, if we consider *generic frameworks* i.e. frameworks in which there are no algebraic dependencies between the coordinates of the vertices.

It is known that the rigidity of frameworks in $\mathbb{R}^d$ is a generic property, that is, the rigidity of (G, p) depends only on the graph G and not the particular realization p, if (G, p) is generic, see [27]. We say that the graph G is *rigid* in $\mathbb{R}^d$ if every (or equivalently, if some) generic realization of G in $\mathbb{R}^d$ is rigid. The problem of characterizing when a graph is rigid in $\mathbb{R}^d$ has been solved for $d = 1, 2$, and is a major open problem for $d \geq 3$.

A similar situation holds for global rigidity. Results of Connelly [3] and Gortler, Healy and Thurston [8] imply that the global rigidity of d-dimensional frameworks is a generic property for all $d \geq 1$. We say that a graph G is *globally rigid* in $\mathbb{R}^d$ if every (or equivalently, if some) generic realization of G in $\mathbb{R}^d$ is globally rigid. Hendrickson [9] proved two key necessary conditions for the global rigidity of a graph. We say that G is *redundantly rigid in* $\mathbb{R}^d$ if removing any edge of G results in a rigid graph.

Theorem 1.1 *([9]). Let G be a globally rigid graph in $\mathbb{R}^d$. Then either G is a complete graph on at most $d+1$ vertices, or G is*