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Notes

Highly connected monochromatic subgraphs of two-colored complete graphs

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ABSTRACT

We show that each 2-edge coloring of the complete graph on n vertices leads to a monochromatic k-connected subgraph on at least $n-2(k-1)$ vertices, provided $n \geq 4k-3$. It settles in the affirmative a conjecture of Bollobás and Gyárfás.

A conjecture of Bollobás and Gyárfás [1] (see also Gyárfás [3]) states that each 2-coloring of the edges of the complete graph on n vertices, where $n \geq 4k-3$, leads to a k-connected monochromatic subgraph on at least $n-2(k-1)$ vertices. In this note we settle it in the affirmative in a slightly stronger version.

Theorem. *Let $k \geq 2$ and $n \geq 4k-3$. Then, in each 2-coloring of the edges of the complete graph on n vertices either one can find a monochromatic k-connected subgraph on more than $n-2(k-1)$ vertices, or there exist monochromatic k-connected graphs on $n-2(k-1)$ vertices in both colors.*

The following example from [1] shows that the inequalities in Theorem cannot be improved. Let us split the vertex set $[n]=\{1,2,\ldots,n\}$ of the complete graph into five

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subsets V_i, $i = 1, 2, \ldots, 5$, where $|V_i| = k - 1$ for $i = 1, 2, 3, 4$, and $|V_5| = n - 4(k - 1)$. Now let us color red the edges joining V_1 and V_2, those joining V_3 and V_4, as well as all the edges with both ends in the set $V_2 \cup V_3 \cup V_5$; all remaining edges we color blue. It is easy to see that the largest monochromatic k-connected subgraphs in this graph contains V_5 and two of the other sets.

Bollobás and Gyárfás [1] verified the conjecture for $k \leq 2$ and observed that it is enough to prove it for $n < 7(k - 1)$. Liu, Morris, and Prince proved it for $k = 3$, and showed that it holds for $n \geq 13k - 15$ (see [4] and [5]). Recently, Fujita and Magnant [2] verified the conjecture for $n \geq 6.5(k - 1)$.

We remark that, as noticed by Bollobás and Gyárfás [1] (see also [3]), the following result is a direct consequence of the theorem.

Corollary. *Each 2-coloring of the complete graph on n vertices leads to a monochromatic $\lceil n/4 \rceil$-connected subgraph.* □

Proof of Theorem. Let us suppose that the assertion of Theorem does not hold. Let $E(R) \cup E(B)$ be a coloring of the edges of the complete graph with vertex set $[n]$, where $n \geq 4k - 3$, with red and blue such that it contains no red k-connected subgraphs on at least $n - 2k + 3$ vertices, no blue k-connected subgraph on at least $n - 2k + 2$ vertices, and the coloring is chosen in such a way that it maximizes the number of red edges. By $R = ([n], E(R))$ and $B = ([n], E(B))$ we denote the red and blue graphs respectively. In order to get a contradiction we state and verify a number of claims.

Claim 1. *R contains at most one maximal $(k - 1)$-connected subgraph with at least $n - 2k + 2$ vertices.*

Proof. Let W_1 and W_2 be vertex sets of two maximal $(k - 1)$-connected subgraphs of R, each of size at least $n - 2k + 2$. We call these subgraphs F_1 and F_2 respectively. Since the subgraph spanned in R by $W_1 \cup W_2$ is not $(k - 1)$-connected, it contains a cut S, $|S| \leq k - 2$. But both F_1 and F_2 are $(k - 1)$-connected, so S separates $W_1 \setminus S$ from $W_2 \setminus S$ and contains all vertices from $W_1 \cap W_2$. But then the sets $W_1 \setminus S$ and $W_2 \setminus S$ span in B the complete bipartite graph F on at least

$$\begin{aligned} |W_1 \cup W_2| - |S| &= |W_1| + |W_2| - |W_1 \cap W_2| - |S| \\ &\geq n - 2k + 2 + n - 2k + 2 - k + 2 - k + 2 \\ &\geq n - 2k + 5 \geq n - 2k + 2 \end{aligned}$$

vertices. Moreover, each set of the bipartition contains at

$$n - 2k + 2 - (k - 2) \geq k + 1,$$

vertices, so F is k-connected contradicting our assumption on the coloring. □