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Notes

Every totally real algebraic integer is a tree eigenvalue

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ABSTRACT

Graph eigenvalues are examples of totally real algebraic integers, i.e. roots of real-rooted monic polynomials with integer coefficients. Conversely, the fact that every totally real algebraic integer occurs as an eigenvalue of some finite graph is a deep and remarkable result, conjectured forty years ago by Hoffman, and proved seventeen years later by Estes. This short paper provides an independent and elementary proof of a stronger statement, namely that the graph may actually be chosen to be a tree. As a by-product, our result implies that the atoms of the limiting spectrum of $n \times n$ symmetric matrices with independent Bernoulli $(\frac{c}{n})$ entries are exactly the totally real algebraic integers. This settles an open problem raised by Ben Arous (2010).

1. Introduction

By definition, the *eigenvalues* of a finite graph $G = (V, E)$ are the roots of the characteristic polynomial $\Phi_G(x) := \det(xI - A)$, where $A = \{A_{i,j}\}_{i,j \in V}$ is the adjacency matrix of G:

$$A_{i,j} = \begin{cases} 1 & \text{if } \{i, j\} \in E \\ 0 & \text{otherwise.} \end{cases}$$

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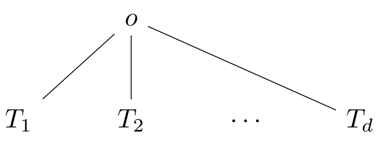
Those eigenvalues capture a considerable amount of information about G. For a detailed account, see e.g. [3,2]. It follows directly from this definition that any graph eigenvalue is a *totally real algebraic integer*, i.e. a root of some real-rooted monic polynomial with integer coefficients. Remarkably enough, the converse is also true: every totally real algebraic integer is an eigenvalue of some finite graph. This deep result was conjectured forty years ago by Hoffman [5], and established seventeen years later by Estes [4], see also [1]. The present paper provides an independent, elementary proof of a stronger statement, namely that the graph may actually be chosen to be a tree.

Theorem 1. *Every totally real algebraic integer is an eigenvalue of some finite tree.*

Trees undoubtedly play a crucial role in many aspects of graph theory. We therefore believe that the strengthening provided by Theorem 1 may be of independent interest, beyond the fact that it provides a simpler proof of Hoffman's conjecture. In addition, Theorem 1 settles an open problem raised by Ben Arous [7, Problem 14], namely that of determining the set of atoms Σ of the limiting spectrum of $n \times n$ symmetric matrices with independent Bernoulli $(\frac{c}{n})$ entries. Indeed, it easily follows from the log-Hölder continuity of the spectrum of integer matrices at algebraic numbers (see e.g. [8, Section 6]) that Σ is contained in the set of totally real algebraic integers. On the other hand, Σ contains all tree eigenvalues, as noted by Ben Arous. Theorem 1 precisely shows that those inner and outer bounds coincide.

2. Outline of the proof

Let T be a finite tree with a distinguished vertex o (the root). Removing o naturally yields a decomposition of T into smaller rooted trees $T_1, \dots, T_d$ ($d \in \mathbb{N}$) as depicted in the following diagram:



To such a rooted tree, let us associate the rational function

$$\mathfrak{f}_T(x) = 1 - \frac{\varPhi_T(x)}{x\varPhi_{T\setminus o}(x)} = 1 - \frac{\varPhi_T(x)}{x\varPhi_{T_1}(x) \cdots \varPhi_{T_d}(x)}. \tag{1}$$

Expressed in terms of this function, the classical recursion for the characteristic polynomial of trees (see e.g. [2, Proposition 5.1.1]) simply reads

$$\mathfrak{f}_T(x) = \frac{1}{x^2} \sum_{i=1}^{d} \frac{1}{1 - \mathfrak{f}_{T_i}(x)}, \tag{2}$$