Journal of Combinatorial Theory, Series B 113 (2015) 1–17

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series B

www.elsevier.com/locate/jctb

Colouring quadrangulations of projective spaces

Tomáš Kaiser [a,1], Matěj Stehlík [b,2]

[a] *Department of Mathematics, Institute for Theoretical Computer Science (CE-ITI), and European Centre of Excellence NTIS (New Technologies for the Information Society), University of West Bohemia, Univerzitní 8, 306 14 Plzeň, Czech Republic*

[b] *UJF-Grenoble 1/CNRS/Grenoble-INP, G-SCOP UMR5272, Grenoble, F-38031, France*

ARTICLE INFO

Article history:
Received 21 October 2013
Available online 31 December 2014

Keywords:
Quadrangulation
Projective space
Chromatic number

ABSTRACT

A graph embedded in a surface with all faces of size 4 is known as a quadrangulation. We extend the definition of quadrangulation to higher dimensions, and prove that any graph G which embeds as a quadrangulation in the real projective space P^n has chromatic number $n + 2$ or higher, unless G is bipartite. For $n = 2$ this was proved by Youngs (1996) [20]. The family of quadrangulations of projective spaces includes all complete graphs, all Mycielski graphs, and certain graphs homomorphic to Schrijver graphs. As a corollary, we obtain a new proof of the Lovász–Kneser theorem.

1. Introduction

A graph which embeds in the real projective plane P^2 so that every face is bounded by a walk of length 4 is called a (2-dimensional) *projective quadrangulation*. A remarkable

E-mail addresses: kaisert@kma.zcu.cz (T. Kaiser), matej.stehlik@g-scop.inpg.fr (M. Stehlík).

[1] The Institute for Theoretical Computer Science (CE-ITI) is supported by project P202/12/G061 of the Czech Science Foundation.

[2] Partially supported by ANR project Stint under reference ANR-13-BS02-0007 and by the LabEx PERSYVAL-Lab (ANR-11-LABX-0025).

http://dx.doi.org/10.1016/j.jctb.2014.12.007
0095-8956/

result of Youngs [20] asserts that the chromatic number of a projective quadrangulation is either 2 or 4. On the last page of [20], Youngs notes:

> ...it would be equally worthwhile to increase the chromatic number [of the graphs in question]. A possible step in this direction is to jump from a (2-dimensional) projective plane to a higher dimensional projective space. This may not be a fruitful path to follow, and the only evidence the author can suggest in its favor is that the 5-chromatic Mycielski graphs embed pleasantly in projective 3-space in a similar fashion to their 4-chromatic counterparts in 2-space.

In this paper, we show that Youngs' intuition was correct as we extend the lower bound in his theorem to the n-dimensional real projective space P^n. To do so, we extend the notion of quadrangulation to higher dimensions as follows (for definitions, see Section 2).

Let K be a generalised simplicial complex (there may be more than one simplex with the same set of vertices, unlike in the usual simplicial complex). A *quadrangulation* of K is a spanning subgraph G of its 1-skeleton $K^{(1)}$ such that every (inclusionwise) maximal simplex of K induces a complete bipartite subgraph of G with at least one edge. If the polyhedron of K is homeomorphic to a topological space X, we say that the natural embedding of G in X is a quadrangulation of X.

Note that if K triangulates the projective plane, then a quadrangulation of K is a projective quadrangulation according to the usual definition recalled at the beginning of this section. Conversely, given a projective quadrangulation H, we can triangulate its faces and obtain H as a quadrangulation of the resulting generalised simplicial complex. More precisely, this is true if none of the faces of H contains a crosscap; otherwise, two edges of H will be doubled in the process. However, this difference between the two definitions is unimportant as long as we are interested in vertex colouring.

Our main result is the following generalisation of the lower bound of Youngs.

Theorem 1.1. *If G is a non-bipartite quadrangulation of the n-dimensional projective space P^n, then $\chi(G) \geq n+2$.*

We show that the family of quadrangulations of projective spaces includes all complete graphs and all (generalised) Mycielski graphs. We also prove the following result about the Schrijver graph $SG(n,k)$. (Recall that a graph G is *homomorphic* to a graph H if there exists a mapping $f : V(G) \to V(H)$ such that $f(u)f(v) \in E(H)$ whenever $uv \in E(G)$; note that, in this case, $\chi(G) \leq \chi(H)$.)

Theorem 1.2. *Let $n > 2k$ and $k \geq 1$. There exists a non-bipartite quadrangulation of P^{n-2k} that is homomorphic to $SG(n,k)$.*

Since the Schrijver graph $SG(n,k)$ is a subgraph of the Kneser graph $KG(n,k)$, Theorems 1.1 and 1.2 give an alternative proof of the Lovász–Kneser theorem [9], namely $\chi(KG(n,k)) \geq n-2k+2$.