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Linear embeddings of graphs and graph limits

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ABSTRACT

Consider a random graph process where vertices are chosen from the interval $[0, 1]$, and edges are chosen independently at random, but so that, for a given vertex x, the probability that there is an edge to a vertex y decreases as the distance between x and y increases. We call this a random graph with a linear embedding.

We define a new graph parameter Γ^*, which aims to measure the similarity of the graph to an instance of a random graph with a linear embedding. For a graph G, $\Gamma^*(G) = 0$ if and only if G is a unit interval graph, and thus a deterministic example of a graph with a linear embedding.

We show that the behaviour of Γ^* is consistent with the notion of convergence as defined in the theory of dense graph limits. In this theory, graph sequences converge to a symmetric, measurable function on $[0, 1]^2$. We define an operator Γ which applies to graph limits, and which assumes the value zero precisely for graph limits that have a linear embedding. We show that, if a graph sequence $\{G_n\}$ converges to a function w, then $\{\Gamma^*(G_n)\}$ converges as well. Moreover, there exists a function w^* arbitrarily close to w under the box distance, so that $\lim_{n\to\infty} \Gamma^*(G_n)$ is arbitrarily close to $\Gamma(w^*)$.

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1. Introduction

Consider the following random graph model on n vertices. Vertices are randomly chosen from the interval $[0, 1]$ according to a given distribution. Then, for each pair of vertices x, y, independently, an edge is added with probability $w(x, y)$, where $w : [0, 1]^2 \to [0, 1]$ is a symmetric, measurable function.

In this article, we are interested in the special case where w is increasing towards the diagonal. Specifically, for $x < y$, $w(x, y)$ decreases as y increases or x decreases. Such a random graph has a linear geometric interpretation: vertices are embedded in the line segment $[0, 1]$, and live in a probability landscape where link probabilities decrease as the linear distance between points increases. We will refer to this as a random graph with a *linear embedding*.

Consider now the problem of recognizing graphs produced by a random graph process with a linear embedding. If the labels of the vertices are provided, this question may be answered by regular statistical methods. When only the isomorphism type of the graph is given, the question becomes more complicated. We address the question of how to recognize graphs whose structure is consistent with that of a random graph with a linear embedding.

Recognition is easy in the special case of *unit interval graphs*, or *one-dimensional geometric graphs*. Here, the selection of vertices is random, but the edge formation is deterministic. In other words, the function w governing edge formation only takes values in $\{0, 1\}$. In this paper, we introduce a graph parameter Γ^* which aims to measure the similarity of the graph to an instance of a random graph with a linear embedding. We show that Γ^* of a given graph equals zero if and only if the graph is a one-dimensional geometric graph (Proposition 3.4). We then consider the behaviour of Γ^* when it is applied to convergent sequences of graphs $\{G_n\}$, where convergence is defined as in the theory of graph limits as developed by Lovász and Szegedy in [20].

In this theory, convergence is defined based on homomorphism densities, and the limit is a symmetric, measurable function. The theory is developed and extended to sequences of random graphs by Borgs et al. in [6,8,7] and is explored further by Lovász and others (see for example [5,9,22]. See also the recent book [19]). As shown by Diaconis and Janson in [14], the theory of graph limits is closely connected to the probabilistic theory of exchangeable arrays. A different view, where the limit object is referred to as a *kernel*, is provided by Bollobás, Janson and Riordan in [1,2]. The connection with the results of Borgs et al. and an extension of the theory to sparse graphs are presented in [4].

Homomorphism densities characterize the isomorphism type of a (twine-free) graph. A graph sequence $\{G_n\}$ converges if and only if all of the homomorphism densities of the graphs G_n converge. Moreover, the limits of all these homomorphism densities can be obtained from a symmetric, measurable function w on $[0, 1]^2$ which represents the "limit object". Thus, w encapsulates the local structure of the graphs in the sequence. Conversely, the randomly growing graph sequence obtained from w, according to the