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Notes

The Erdős–Hajnal conjecture for paths and antipaths

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ABSTRACT

We prove that for every k, there exists $c_k > 0$ such that every graph G on n vertices with no induced path P_k or its complement $\overline{P_k}$ contains a clique or a stable set of size n^{c_k}.

An *n-graph* is a graph on n vertices. For every vertex x, $N(x)$ denotes the neighborhood of x, that is the set of vertices y such that xy is an edge. The degree $\deg(x)$ is the size of $N(x)$. In this note, we only consider classes of graphs that are closed under induced subgraphs. Moreover a class $\mathcal{C}$ is *strict* if it does not contain all graphs. It is said to have the (*weak*) *Erdős–Hajnal property* if there exists some $c > 0$ such that every graph of $\mathcal{C}$ contains a clique or a stable set of size n^c where n is the size of G. The Erdős–Hajnal conjecture [8] asserts that every strict class of graphs has the Erdős–Hajnal property; see [3] for a survey. This fascinating question is open even for graphs not inducing a cycle of length five. When excluding a single graph H, Alon, Pach and Solymosi showed

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in [2] that it suffices to consider *prime* H, namely graphs without nontrivial modules (a *module* is a subset V' of vertices such that for every $x, y \in V'$, $N(x) \setminus V' = N(y) \setminus V'$). A natural approach is then to study classes of graphs with intermediate difficulty, hoping to get a proof scheme which could be extended. A natural prime candidate to forbid is certainly the path. Unfortunately, even excluding the path on five vertices seems already hard. Chudnovsky and Zwols studied the class $\mathcal{C}_k$ of graphs not inducing the path P_k on k vertices or its complement $\overline{P_k}$. They proved the Erdős–Hajnal property for P_5 and $\overline{P_6}$-free graphs [7]. This was extended for P_5 and $\overline{P_7}$-free graphs by Chudnovsky and Seymour [6]. Moreover structural results have been provided for $\mathcal{C}_5$ [5,4]. We show in this note that for every fixed k, the class $\mathcal{C}_k$ has the Erdős–Hajnal property. An n-graph is an *ε-stable set* if it has at most $\varepsilon\binom{n}{2}$ edges. The complement of an ε-stable set is an *ε-clique*. Fox and Sudakov [11] proved the following:

Theorem 1. *(See [11].) For every positive integer k and every $\varepsilon \in (0, 1/2)$, there exists $\delta > 0$ such that every n-graph G satisfies one of the following:*

- *G induces all graphs on k vertices.*
- *G contains an ε-stable set of size at least δn.*
- *G contains an ε-clique of size at least δn.*

Note that a stronger result was previously showed by Rödl [14] using Szemerédi's regularity lemma, but Fox and Sudakov's proof provides a much better quantitative estimate ($\delta = 2^{-ck(\log 1/\varepsilon)^2}$ for some constant c). They further conjecture that a polynomial estimate should hold, which would imply the Erdős–Hajnal conjecture.

In a graph G, a *biclique of size t* is a (not necessarily induced) complete bipartite subgraph (X, Y) such that both $|X|, |Y| \geq t$. Observe that it does not require any condition inside X or inside Y. Erdős, Hajnal and Pach proved in [9] that for every strict class $\mathcal{C}$, there exists some $c > 0$ such that for every n-graph G in $\mathcal{C}$, G or its complement $\overline{G}$ contains a biclique of size n^c. This "half" version of the conjecture was improved to a "three quarter" version by Fox and Sudakov [12], where they show the existence of a polynomial size stable set or biclique. Following the notation of [10], a class $\mathcal{C}$ of graphs has the *strong Erdős–Hajnal property* if there exists a constant c such that for every n-graph G in $\mathcal{C}$, G or $\overline{G}$ contains a biclique of size cn. It was proved that having the strong Erdős–Hajnal property implies having the (weak) Erdős–Hajnal property:

Theorem 2. *(See [1,10].) If $\mathcal{C}$ is a class of graphs having the strong Erdős–Hajnal property, then $\mathcal{C}$ has the weak Erdős–Hajnal property.*

Proof (sketch). Let c be the constant of the strong Erdős–Hajnal property, meaning that for every n-graph G in $\mathcal{C}$, G or $\overline{G}$ contains a biclique of size cn. Let $c' > 0$ be such that $c^{c'} \geq 1/2$. We prove by induction that every n-graph G in $\mathcal{C}$ induces a P_4-free graph of size $n^{c'}$. By our hypothesis on $\mathcal{C}$, there exists, say, a biclique (X, Y) of size cn in G.