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Circular flow number of highly edge connected signed graphs

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A R T I C L E I N F O

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This paper proves that for any positive integer k, every essentially $(2k+1)$-unbalanced $(12k-1)$-edge connected signed graph has circular flow number at most $2 + \frac{1}{k}$.

1. Introduction

Suppose G is a graph. A *circulation* in G is an orientation D of G together with a mapping $f : E(G) \to \mathbb{R}$. A circulation in G can be denoted by a pair (D, f). However, for simplicity, we usually denote it by f, and call D the *orientation associated with* f. The *boundary* of a circulation f is the map $\partial f : V(G) \to \mathbb{R}$ defined as

$$\partial f(v) = \sum_{e \in E_D^+(v)} f(e) - \sum_{e \in E_D^-(v)} f(e). \tag{1}$$

Here $E_D^+(v)$ (resp. $E_D^-(v)$) is the set of directed edges in D of the form (v, u) (resp. of the form (u, v)).

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A *flow in* G is a circulation in G with $\partial f = 0$. If r is a real number and f is a flow with $1 \le |f(e)| \le r - 1$ for every edge e, then f is called a *circular* r*-flow* in G. The *circular flow number* $\Phi_c(G)$ of G is the least r such that G admits a circular r-flow. If r is an integer and $1 \le |f(e)| \le r - 1$ are integers then f is called a *nowhere zero* r*-flow*. The *flow number* $\Phi(G)$ of G is the least integer r such that G admits a nowhere zero r-flow. It is known [2] that $\Phi(G) = \lceil \Phi_c(G) \rceil$ for any bridgeless graph G.

Integer flow was originally introduced by Tutte [9,10] as a generalization of map colouring. Tutte proposed the following three conjectures that motivated most of the studies on integer flow in graphs.

- **5-flow conjecture**: Every bridgeless graph admits a nowhere zero 5-flow.
- **4-flow conjecture**: Every bridgeless graph with no Petersen minor admits a nowhere zero 4-flow.
- **3-flow conjecture**: Every 4-edge connected graph admits a nowhere zero 3-flow.

The concept of circular flow number was introduced in [2] in 1998 as the dual of the circular chromatic number (cf. [13,14]), and as a refinement of the flow number. However, in early 1980s, Jaeger [3] already studied circular flow in graphs and proposed a conjecture which is equivalent to the following:

- $\mathbf{(2 + \frac{1}{k})}$**-flow conjecture**: For any positive integer k, if G is $4k$-edge connected, then $\Phi_c(G) \le 2 + \frac{1}{k}$.

Jaeger's conjecture is very strong. The $k = 1$ case is the 3-flow conjecture, and the $k = 2$ case implies the 5-flow conjecture.

All the above conjectures remain open. Recently, Thomassen [8] made a breakthrough in the study of 3-flow conjecture by proving that every 8-edge connected graph admits a nowhere zero 3-flow. Moreover, for $k \ge 1$, every $(8k^2 + 10k + 3)$-edge connected graph has circular flow number at most $2 + \frac{1}{k}$. This result is improved by Lovász, Thomassen, Wu and Zhang in [5], where it is proved that for any positive integer k, if a graph G has odd edge connectivity at least $6k + 1$, then $\Phi_c(G) \le 2 + \frac{1}{k}$.

This paper proves an analog of this result for signed graphs.

A *signed graph* is a pair (G, σ), where G is a graph and $\sigma : E(G) \to \{1, -1\}$ assigns to each edge a sign: an edge e is either a *positive edge* (i.e., $\sigma(e) = 1$) or a *negative edge* (i.e., $\sigma(e) = -1$). An *orientation* τ of (G, σ) assigns "orientations" to the edges of G as follows: if $e = xy$ is a positive edge, then the edge is oriented either from x to y or from y to x. In the former case, $e \in E_\tau^+(x) \cap E_\tau^-(y)$, and in the later case, $e \in E_\tau^-(x) \cap E_\tau^+(y)$. If $e = xy$ is a negative edge, then the edge is oriented either from both x and y or towards both x and y. In the former case, $e \in E_\tau^+(x) \cap E_\tau^+(y)$ and e is called a *sink edge*. In the later case, $e \in E_\tau^-(x) \cap E_\tau^-(y)$ and e is called a *source edge*. An orientation τ of (G, σ) may be viewed as a mapping which assigns to each positive edge one of its end vertices as the *head* of the directed edge, and labels each negative edge either as a source edge