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# Edge-primitive tetravalent graphs

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A B S T R A C T

A graph is edge-primitive if its automorphism group acts primitively on edges. In 1973 Weiss [28] determined edge-primitive cubic graphs. In this paper, we classify edge-primitive tetravalent graphs.



## 1. Introduction

Throughout this paper, we consider undirected finite graphs without loops or multiple edges. For a graph $X$, we use $V(X)$, $E(X)$ and $\mathrm{Aut}(X)$ to denote its vertex set, edge set, and its automorphism group, respectively. For $u, v \in V(X)$, $\{u, v\}$ is the edge incident to $u$ and $v$ in $X$, and $X_1(v)$ is the *neighborhood* of $v$ in $X$.

A graph $X$ is *$G$-edge-primitive* or simply *edge-primitive* if $G \leq \mathrm{Aut}(X)$ acts primitively on $E(X)$. A graph $X$ is said to be *$G$-vertex-transitive* if $G \leq \mathrm{Aut}(X)$ acts transitively on $V(X)$. An $s$-arc in a graph $X$ is an ordered $(s+1)$-tuple $(v_0, v_1, \cdots, v_{s-1}, v_s)$ of vertices of $X$ such that $v_{i-1}$ is adjacent to $v_i$ for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. A 0-arc is a vertex and a 1-arc is also called an *arc* for short. A graph $X$ is said to be

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**Table 1**
$s$-transitive edge-primitive tetravalent graphs.

| $s$ | $G$ | $G_v$ | $G_e$ | Comments |
|---|---|---|---|---|
| 2 | $S_5$ | $S_4$ | $D_{12}$ | complete graph $K_5$ |
| 2 | $PGL(2,7)$ | $S_4$ | $D_{12}$ | co-Heawood graph |
| 3 | $S_4 \operatorname{wr} S_2$ | $S_4 \times S_3$ | $S_3 \operatorname{wr} S_2$ | complete bipartite graph $K_{4,4}$ |
| 4 | $PSL(3,3).\mathbb{Z}_2$ | $\mathbb{Z}_3^2 \rtimes GL(2,3)$ | $3_+^{1+2}{:}D_8$ | $PSL(3,3).\mathbb{Z}_2$-graph |
| 4 | $M_{12}.\mathbb{Z}_2$ | $\mathbb{Z}_3^2 \rtimes GL(2,3)$ | $3_+^{1+2}{:}D_8$ | $M_{12}.\mathbb{Z}_2$-graph |
| 7 | $G_2(3).\mathbb{Z}_2$ | $[3^5] \rtimes GL(2,3)$ | $(3^2.(3 \times 3_+^{1+2})){:}D_8$ | $G_2(3).\mathbb{Z}_2$-graph |

$(G,s)$-*arc-transitive* if $G \le \mathrm{Aut}(X)$ is transitive on the set of $s$-arcs in $X$. A $(G,s)$-arc-transitive graph is said to be $(G,s)$-*transitive* if it is not $(G,s+1)$-arc-transitive. A graph $X$ is said to be $s$-*arc-transitive* and $s$-*transitive* if $X$ is $(\mathrm{Aut}(X),s)$-arc-transitive and $(\mathrm{Aut}(X),s)$-transitive, respectively.

In 1973 Weiss [28] determined all cubic edge-primitive graphs, which are the complete bipartite graph $K_{3,3}$, the Heawood graph of order 14, the Biggs–Smith cubic distance-transitive graph of order 102 and the Tutte–Coxeter graph of order 30 (also known as Tutte's 8-cage or the Levi graph). Recently, Giudici and Li [8] systematically analyzed edge-primitive graphs via the O'Nan-Scott Theorem to determine the possible edge and vertex actions of such graphs, and determined all $G$-edge-primitive graphs for $G$ an almost simple group with socle $PSL(2,q)$, where $q$ is a prime power and $q \ne 2$ or 3. The main result of this paper is to present a classification of edge-primitive tetravalent graphs.

In what follows we denote by $\mathbb{Z}_n$ the cyclic group of order $n$, by $\mathbb{Z}_p^n$ the elementary abelian group of order $p^n$ ($p$ a prime), by $D_{2n}$ the dihedral group of order $2n$, by $Q_{4n}$ the generalized quaternion group of order $4n$, and by $A_n$ and $S_n$ the alternating group and the symmetric group of degree $n$. For two subgroups $M$ and $N$ of a group $G$, $N \rtimes M$ (or $N{:}M$) stands for a semidirect product of $N$ by $M$, $N.M$ stands for an extension of $N$ by $M$, and $N \operatorname{wr} M$ stands for the wreath product of $N$ by $M$ when $M$ is a permutation group (see for example [5]).

**Theorem 1.1.** *Let $X$ be an edge-primitive tetravalent graph, and let $G = \mathrm{Aut}(X)$. Then $X$ is $s$-transitive with $s \ge 2$, and for a vertex $v$ and an edge $e$, $s$, $G$, $G_v$ and $G_e$ are listed in Table 1.*

**Remark.** For a positive integer $n$, $[n]$ and $n$ stand for an arbitrary group and a cyclic group of order $n$ respectively, and $3_+^{1+2}$ is the unique non-abelian group of order 27 with exponent 3. The co-Heawood graph is the complement graph of the Heawood graph with respect to the complete bipartite graph $K_{7,7}$ (see [8, Example 2.2]). The $PSL(3,3).\mathbb{Z}_2$-graph and $G_2(3).\mathbb{Z}_2$-graph are the classical generalized polygon graphs (see [19, Section 2]), which are coset graphs on $PSL(3,3).\mathbb{Z}_2$ and $G_2(3).\mathbb{Z}_2$ respectively. The $M_{12}.\mathbb{Z}_2$-graph is a coset graph on $M_{12}.\mathbb{Z}_2$, which was defined by Weiss [29].