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Tournament minors

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ABSTRACT

We say a digraph G is a *minor* of a digraph H if G can be obtained from a subdigraph of H by repeatedly contracting a strongly-connected subdigraph to a vertex. Here, we show that the class of all tournaments is a well-quasi-order under minor containment.

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1. Introduction

The "minor" relation for graphs is well-established, but how it should be extended to digraphs is not clear. In digraphs, contracting an edge may yield a directed cycle, even starting from an acyclic digraph, and this seems undesirable for a theory of excluded minors. One way to avoid this is to permit the contraction only of certain special edges; for example, if an edge uv is the only edge with tail u or the only edge with head v, then contracting uv does not yield a new directed cycle (see for instance [4]). Another way, too complicated to explain here, is discussed for instance in [5].

A third way to extend minors of graphs to digraphs is as follows. For graphs, one can define contraction in terms of contracting edges, or in terms of contracting connected subgraphs, and it comes to the same thing. But for digraphs, contracting edges and contracting strongly-connected subdigraphs lead to different minor relations, and in this

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paper we study the second. (A digraph G is *strongly-connected* if G is non-null and there exists a directed path from u to v for every $u, v \in V(G)$.) We say a digraph H is a *minor* of a digraph G if H can be obtained from a subdigraph of G by repeatedly contracting a strongly-connected subdigraph to a vertex. (Note that we do not create "new" directed cycles after contracting a strongly-connected subdigraph.) Equivalently, a digraph H is a *minor* of a digraph G if there exists a mapping ϕ defined on $V(H)$ such that:

- for every $v \in V(H)$, $\phi(v)$ is a non-null strongly-connected subdigraph of G;
- if $u, v \in V(H)$ and $u \neq v$, then $\phi(u)$ and $\phi(v)$ are vertex-disjoint; and
- for every $u, v \in V(H)$ (not necessarily distinct), if there are k edges in H with tail u and head v, then there are at least k edges in G with head in $V(\phi(u))$ and tail in $V(\phi(v))$, and not contained in $E(\phi(x))$ for any $x \in V(H)$.

We call such a map ϕ a *model* of H in G.

We first give some definitions. Every digraph in this paper is finite. We say a digraph G is *simple* if it is loopless and there is at most one edge $uv \in E(G)$ for all distinct $u, v \in V(G)$. A simple digraph G is *semi-complete* if either $uv \in E(G)$ or $vu \in E(G)$ for all distinct $u, v \in V(G)$. A semi-complete digraph G is a *tournament* if exactly one of uv and vu is an edge of G for all distinct $u, v \in V(G)$.

An important property of minors for graphs is that they define a "well-quasi-order" [7]. A *quasi-order* $Q = (V(Q), \leq_Q)$ consists of a class $V(Q)$ and a reflexive transitive relation $\leq_Q$ on $V(Q)$. A quasi-order Q is called a *well-quasi-order* or *wqo* if for every infinite sequence $q_1, q_2, \ldots$ of elements of $V(Q)$, there exist $j > i \geq 1$ such that $q_i \leq_Q q_j$. Neil Robertson and the second author proved that the class of all graphs is a wqo under the minor relation in [7].

The analogous statement is not true for directed minors. For example, a directed cycle is not a minor of a bigger directed cycle, and so if we take an infinite set of digraphs, all directed cycles of different lengths, then this set is an infinite antichain under the minor order. However, what if we consider some subclass, say the class of all tournaments? (The subdigraph relation does not define a wqo even for the class of all tournaments. We leave finding a counterexample as an exercise for the reader.)

In this paper, we prove that minor containment defines a wqo for the class of all semi-complete digraphs, and therefore the same is true for the class of all tournaments. We also prove it is not a wqo for some closely-related classes.

1.1. *The class of all semi-complete digraphs is a wqo under minor containment.*

In [1], Maria Chudnovsky and the second author proved that the class of all semi-complete digraphs is a wqo under "immersion", by using a digraph parameter called "cut-width". Here, we prove the analogous statement for minors by using another parameter called *path-width*. Path-width for undirected graphs was introduced in [6], and it has a natural extension to digraphs, discussed for instance in [2].

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