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Regular maps with simple underlying graphs

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ABSTRACT

A regular map is a symmetric embedding of a graph (or multigraph) on some closed surface. In this paper we consider the genus spectrum for such maps on orientable surfaces, with simple underlying graph. It is known that for some positive integers g, there is no orientably-regular map of genus g for which both the map and its dual have simple underlying graph, and also that for some g, there is no such map (with simple underlying graph) that is reflexible. We show that for over 83% of all positive integers g, there exists at least one orientably-regular map of genus g with simple underlying graph, and conjecture that there exists at least one for every positive integer g.

1. Introduction

Regular maps are highly symmetric embeddings of graphs or multigraphs on closed surfaces. They generalise the Platonic solids (when these are viewed as embeddings of their 1-skeletons on the sphere) and the regular triangulations, quadrangulations and hexagonal tilings of the torus, to orientable surfaces of higher genus, and to non-orientable surfaces as well.

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The formal study of regular maps was initiated by Brahana [2] in the 1920s and continued by Coxeter (see [8]) and others decades later. Deep connections exist between regular maps and other branches of mathematics, including hyperbolic geometry, Riemann surfaces and, rather surprisingly, number fields and Galois theory. See some of the references at the end of this paper for further background.

Regular maps on the sphere and the torus and other orientable surfaces of small genus are now quite well understood, but until recently, the situation for surfaces of higher genus was something of a mystery. A significant step towards answering some long-standing questions about the genera of orientable surfaces carrying a regular map having no multiple edges, or an 'orientably-regular' map that is chiral (admitting no reflectional symmetry) was taken by Conder, Širáň and Tucker in [7], after the first author noticed patterns in computational data about regular maps of small genus (see [3] and the associated lists of maps available on the first author's website).

One question of interest has been the genus spectrum of orientably-regular maps with simple underlying graph — that is, where the embedded graph has no loops or multiple edges. It is well known that for every $g > 0$ there exists a reflexible regular map of type $\{4g, 4g\}$ on an orientable surface of genus g (with dihedral automorphism group). It follows that there are no 'gaps' in the genus spectrum of orientable surfaces carrying reflexible regular maps. On the other hand, the underlying graphs for these maps are highly degenerate, being bouquets of $2g$ loops based at a single vertex.

A closely-related question concerns the genera of those orientably-regular maps with the property that the underlying graphs of both the map and its dual are simple. From the evidence described in [3], it was discovered that there are gaps in this spectrum: there are no such maps of genus 20, 23, 24, 30, 38, 39, 44, 47, 48, 54, 60, 67, 68, 77, 79, 80, 84, 86, 88 or 95, but there is at least one of genus g for every other g in the range $0 \le g \le 101$.

Two of the main results of [7] were that (a) If M is an orientably-regular but chiral map of genus $p+1$, where p is prime, and $p-1$ is not divisible by 5 or 8, then either M or its topological dual M^* has multiple edges, and (b) if M is a reflexible regular map of genus $p+1$, where p is prime and $p > 13$, then either M or M^* has multiple edges, and if also $p \equiv 1 \bmod 6$, then both M and M^* have multiple edges.

It follows from these that if $g = p + 1$ for some prime $p > 13$ such that $p - 1$ is not divisible by 5 or 8, then there exists no orientably-regular map of genus g such that the underlying graphs of both the map and its dual are simple. Hence there are infinitely many exceptions, well beyond the brief list given two paragraphs above.

On the other hand, if we are happy for just one of M and M^* to have simple underlying graph, then the situation is intriguing. The exceptions arising from (b) for reflexible regular maps are genera of the form $g = p + 1$ where p is a prime congruent to $1 \bmod 6$, but for each of these, there is an orientably-regular but chiral map of type $\{6, 6\}$ of genus g with simple underlying graph. Hence these exceptions for reflexible maps are not exceptions for chiral maps.