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Edge-signed graphs with smallest eigenvalue greater than −2

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ABSTRACT

We give a structural classification of edge-signed graphs with smallest eigenvalue greater than −2. We prove a conjecture of Hoffman about the smallest eigenvalue of the line graph of a tree that was stated in the 1970s. Furthermore, we prove a more general result extending Hoffman's original statement to all edge-signed graphs with smallest eigenvalue greater than −2. Our results give a classification of the special graphs of fat Hoffman graphs with smallest eigenvalue greater than −3.

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1. Introduction

The (adjacency) eigenvalues of a graph G on m vertices are defined as the eigenvalues of its adjacency matrix A. Since A is a real symmetric matrix, its eigenvalues $\lambda_i(A)$ are real; we arrange them as follows

$$\lambda_1(A) \leqslant \lambda_2(A) \leqslant \cdots \leqslant \lambda_m(A).$$

For convenience we will sometimes also refer to each $\lambda_i(A)$ as $\lambda_i(G)$. Much attention has been directed towards the study of graphs with smallest eigenvalue at least -2, see for example [3,6,7,15,19]. Most of this attention has centred around the beautiful theorem of Cameron, Goethals, Shult, and Seidel [4], which classifies graphs having smallest eigenvalue at least -2. In the late 1970s Hoffman [11] studied graphs G with $\lambda_1(G) \geqslant -1-\sqrt{2}$ and later Woo and Neumaier [20] furthered Hoffman's work, introducing the so-called Hoffman graph. Recently, Jang, Koolen, Munemasa, and Taniguchi [14] proposed a programme to classify fat Hoffman graphs with smallest eigenvalue at least -3. The present work fills a part of this programme, and includes the results of [17].

In this article we classify, up to switching equivalence, edge-signed graphs with smallest eigenvalue greater than -2. (See Section 2 for the definition of the eigenvalues of an edge-signed graph.) In particular, we recover as a special case the classification of graphs with smallest eigenvalue greater than -2 given earlier by Doob and Cvetković [7]. As an application, we classify the special graphs of fat Hoffman graphs with smallest eigenvalue greater than -3. Some of such graphs are related to the modified adjacency matrix that appeared in a paper of Hoffman [10]. Below we describe the conjecture Hoffman made about these modified adjacency matrices.

Let T be a tree on $m \geqslant 2$ vertices with line graph $\mathfrak{L}(T)$ and let e be an end-edge of T (one of whose vertices has valency 1). Then e is a vertex of $\mathfrak{L}(T)$. For a graph G and a vertex $v \in V(G)$, define $\hat{A}(G, v)$ to be the adjacency matrix of G, modified by putting a -1 in the diagonal position corresponding to v. In one of his papers [10], Hoffman conjectured that $\lambda_1(\hat{A}(\mathfrak{L}(T), e)) < \lambda_1(\mathfrak{L}(T))$ for all trees T and end-edges e. In Section 4 we prove this conjecture, which we record as the following theorem.

Theorem 1. *Let T be a tree and let e be an end-edge of T. Then $\lambda_1(\hat{A}(\mathfrak{L}(T), e)) < \lambda_1(\mathfrak{L}(T))$.*

Furthermore, using the classification of edge-signed graphs (see Theorem 6) with smallest eigenvalue greater than -2, we prove a generalised version of Hoffman's conjecture (see Theorem 15).

In Section 2 we give our preliminaries. In Section 3 we prove the main part of the classification theorem of edge-signed graphs with smallest eigenvalue greater than -2 leaving the exceptional case to Section 5. In Section 4 we prove Theorem 1 and its generalised version, and in Section 6 we comment on the application to Hoffman graphs with smallest eigenvalue greater than -3.